A small tour of optimization models

Theory of games and markets with examples

Jonas Mockus

Institute of mathematics and Informatics

Kaunas technological university

Lithuania

The idea

"The best way to learn is to do;
the worst way to teach is to talk.
The best way to teach
is to make students ask, and do.
Do not preach facts - stimulate to act".

P. HALMOS, The problem of learning to teach,
Amer. Math. Monthly 82
1975, 750-758

OPTIMIZATION AND APPLICATIONS

Consulting:
jonas2@optimum2.mii.lt
Web-sites:
$http://soften.ktu.lt/\tilde{}mockus$
$http://pilis.if.ktu.lt/\tilde{}jmockus$
$http://eta.ktl.mii.lt/\ mockus$
$http://mockus.us/optimum$ (short)
Textbook (in Lithuanian):
A. Žilinskas
"Matematinis Programavimas"
Kaunas, VDU, 2000

Optimality

Objective function

$$f(x),\ x = (x_1, ..., x_m). \tag{1}$$

Global minimum

$$f(x_A) \leq f(x)\ x \in A, \tag{2}$$

or

$$x_A = \arg\min_A f(x). \tag{3}$$

Here A is a feasible region.

Local minimum

$$f(x_\epsilon) \leq f(x)\ x \in \epsilon. \tag{4}$$

Here ϵ is a vicinity of x_ϵ.

Discrete and convex optimization

Discrete optimization

if variables x_i are discrete.

Linear programming

if

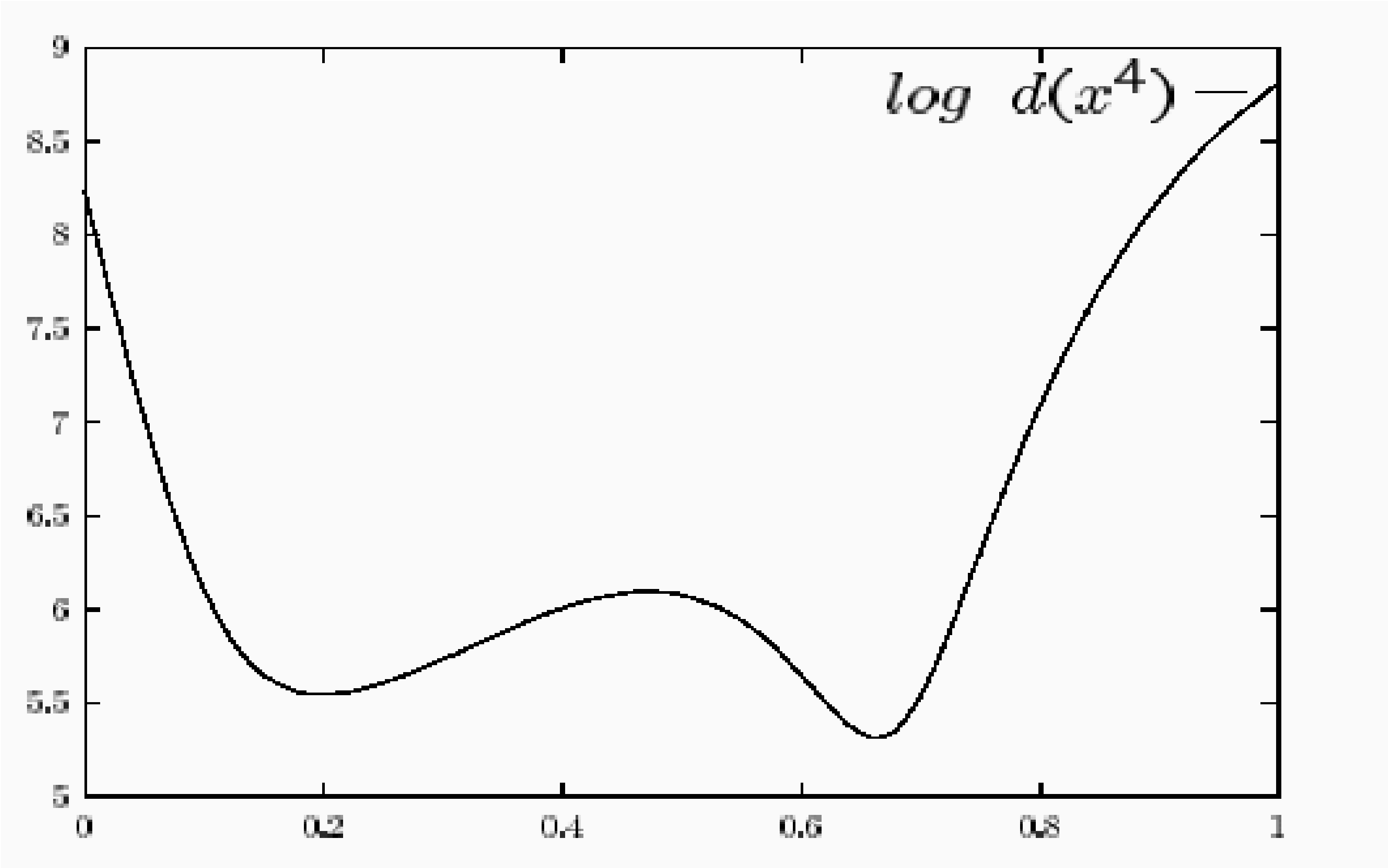
$$f(x) = \sum_{i=1}^{m} c_i x_i, \quad (5)$$

$$A: \sum_{i=1}^{m} a_{ij} x_i \geq b_j,\ j = 1, ..., m,\ x_i \geq\ 0. \quad (6)$$

Convex programming

if both objective $f(x)$ and feasible region A are convex.

Fig. 1: Global and local minima



Global and local minima

Local minimum $x = 0.2$,
global minimum $x = 0.66$.
Function $f(x)$: - multi-modal in [0.0,1.0],
- convex in [0.0, 0.3], [0.6,0.8],
- uni-modal in [0.0,0.48], [0.48,1.0].

Theory of games and markets

Search for equilibrium

A contract with no incentives to brake is equilibrium.
Examples: models of competition, inspection and duel.

Prediction

Examples:
- rates of currency and stocks,
- calls of call-center. **Optimal investment**

Examples:
- portfolio problem,
- optimal insurance.

Discrete programming

Optimal scheduling

Examples:

- flow-shop,
- school schedule.

Sequential decisions

Examples:

- bride,
- buy-a-PC,
- buy-a-car.

Nash equilibrium

Competing servers, Nash version

- Profit of server i:

$$u_i = a_i y_i - x_i, \; i = 1, ..., m, \quad (7)$$

$$\sum_{i=0}^{m} a_i = a, \quad (8)$$

where a rate of customers.

- Customer goes to server i^*, if

$$h_{i^*} \leq h_i, \; i = 0, ..., m, \quad (9)$$

$$h_i = y_i + \gamma_i, \; \gamma_i = n_i / x_i. \quad (10)$$

- y_i service price, γ_i cost of waiting.

Optimal contract

- A contract is optimal if no incentives to break.
  Denote the contract vector $(\bar{y}_i, \bar{x}_i)$. Then the "no-contract vector":

$$(\bar{\bar{y}}_i, \bar{\bar{x}}_i) = \arg \max_{(y_i, x_i)} u_i(y_i, x_i, \bar{y}_j, \bar{x}_j,\ j \neq i), \qquad (11)$$

- Contract $z = (\bar{y}_i, \bar{x}_i,\ i = 1, ..., m)$ is stable if

$$\min_z f(z) = 0, \qquad (12)$$

$$f(z) = \sum_{i=1}^{m} (u_i(\bar{\bar{y}}_i, \bar{\bar{x}}_i) - u_i(\bar{y}_i, \bar{x}_i)). \qquad (13)$$

Stable coalitions

Coalition is stable if no incentives to abandon, $m = 3$.
S_1 is "coalition" of single server,
S_2 is coalition of two servers,
S_3 is monopole- coalition of all three servers.
The profit is divided equally
$u_i(S_3) = (u_i + u_j + u_k)/3$,
$u_i(S_2) = (u_i + u_j)/2$,
$u_i(S_1) = u_i$.
Monopole is stable if
$u_i(S_3) \geq \max\{u_i(S_2), u_i(S_1)\}$.
Free individual competition is stable if
$u_i(S_1) \geq \max\{u_i(S_2), u_i(S_3)\}$.

Walras problem

Denote $y = (y_i,\ i = 1, ..., m),\ p = (p_i,\ i = 1, ..., m),\ x = (x_{ij},\ i, j = 1, ..., m)$. Here x_{ii} are local resources, x_{ij} are server j resources used by i. Profit of i

$$u_i(y, p, x) = a_i y_i + p_i \sum_{j \neq i} x_{ji} - \sum_{j \neq i} p_j x_{ij}, \qquad (14)$$

Customer expences

$$h_i = y_i + \gamma_i,\ \gamma_i = n_i / w_i,\ w_i = c_{0i}(1 - e^{-c_{ii} x_{ii} - c_{ij} x_{ij}}), \qquad (15)$$

where w_i is capacity of server i, b_i is resource of server i, c_{ij} defines efficiency of resources, here local resources x_{ii} are defined by balance condition:

$$x_{ii} + \sum_j x_{ij} = b_i,\ i = 1, ..., m, \qquad (16)$$

Optimal contract in resources x, m=2

Denote the contract resources as $\bar{x}_{12}$ and $\bar{x}_{21}$. Then "no-contract" resources:

$$\bar{\bar{x}}_{12}(y,p) = \arg\max_{x_{12}} u_1(y,p,x_{12},\bar{x}_{21}) \quad (17)$$

$$\bar{\bar{x}}_{21}(y,p) = \arg\max_{x_{21}} u_2(y,p,x_{21},\bar{x}_{12}) \quad (18)$$

Denote

$$\begin{aligned} f_x(y,p,\bar{x}_{12},\bar{x}_{21}) &= u_1(y,p,\bar{\bar{x}}_{12},\bar{x}_{21}) - u_1(y,p,\bar{x}_{12},\bar{x}_{21}) \\ &\quad + u_2(y,p,\bar{\bar{x}}_{21},\bar{x}_{12}) - u_2(y,p,\bar{x}_{21},\bar{x}_{12}) \end{aligned} \quad (19)$$

Contract $(\bar{x}_{12}(y,p),\bar{x}_{21}(y,p))$ is stable if

$$\min_{\bar{x}_{12},\bar{x}_{21}} f_x(y,p,\bar{x}_{12},\bar{x}_{21}) = 0, \quad (20)$$

Optimal resources: $x^\star_{12} = x_{12}(y,p)$ and $x^\star_{21} = x_{21}(y,p)$.

Optimal contract in prices (y,p), m=2

Denote the contract prices $(\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2)$. Then the "no-contract" prices:

$$(\bar{\bar{y}}_1, \bar{\bar{p}}_1) = \arg\max_{(y_1,p_1)} u_1(y_1, \bar{y}_2, p_1, \bar{p}_2, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(y_1, p_1)), \quad (21)$$

$$(\bar{\bar{y}}_2, \bar{\bar{p}}_2) = \arg\max_{(y_2,p_2)} u_2(y_2, \bar{y}_1, p_2, \bar{p}_1, x_{12}(y_2, p_2), x_{21}(\bar{y}_1, \bar{p}_1)) \quad (22)$$

$$\min_{\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2} f(\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2) = 0, \quad (23)$$

$$\begin{aligned} f(\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2) = \quad (24) \\ u_1(\bar{\bar{y}}_1, \bar{y}_2, \bar{\bar{p}}_1, \bar{p}_2, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(\bar{\bar{y}}_1, \bar{\bar{p}}_1)) - \\ u_1(\bar{y}_1, \bar{y}_2, \bar{p}_1, \bar{p}_2, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(\bar{y}_1, \bar{p}_1)) + \\ u_2(\bar{\bar{y}}_2, \bar{y}_1, \bar{\bar{p}}_2, \bar{p}_1, x_{12}(\bar{\bar{y}}_2, \bar{\bar{p}}_2), x_{21}(\bar{y}_1, \bar{p}_1) - \\ u_2(\bar{y}_2, \bar{y}_1, \bar{p}_2, \bar{p}_1, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(\bar{y}_1, \bar{p}_1). \end{aligned}$$

Walras problem, graph error

Relation of profit $u_1(p_1)$ to resource price p_1
(other variables are fixed as the contract prices $(\bar{y_1}, \bar{y_2}, \bar{p_2})$):

The correct relation:

$$u_1(p_1) = a_1\bar{y_1} + p_1 x_{21}(\bar{y_1}, \bar{y_2}, p_1, \bar{p_2}) - p_2 x_{12}(\bar{y_1}, \bar{y_2}, p_1, \bar{p_2}), \quad (25)$$

The observed error:

$$u_1(p_1) = a_1\bar{y_1} + p_1 x_{21}(\bar{y_1}, \bar{y_2}, \bar{p_1}, \bar{p_2}) - p_2 x_{12}(\bar{y_1}, \bar{y_2}, \bar{p_1}, \bar{p_2}), \quad (26)$$

(the same error is in $u_2(p_2)$).

Walras problem, optimization error

Denote the contract prices $(\bar{y_1}, \bar{p_1})$.

The correct "no-contract" prices:

$$(\bar{\bar{y_1}}, \bar{\bar{p_1}}) = \arg\max_{(y_1,p_1)} (a_1 y_1 + p_1 x_{21}(y_1, \bar{y_2}, p_1, \bar{p_2}) - \bar{p_2} x_{12}(y_1, \bar{y_2}, p_1, \bar{p_2})), \quad (27)$$

The suspected error:

$$(\bar{\bar{y_1}}, \bar{\bar{p_1}}) = \arg\max_{(y_1,p_1)} (a_1 y_1 + p_1 x_{21}(y_1, \bar{y_2}, \bar{p_1}, \bar{p_2}) - \bar{p_2} x_{12}(y_1, \bar{y_2}, \bar{p_1}, \bar{p_2})), \quad (28)$$

(the same error is suspected in $(\bar{y_2}, \bar{p_2})$).

Inspector's problem, simple

Inspector's payoff

$$u(i,j) = \begin{cases} p_i g_i q_j, & if\ i = j, \\ 0, & if\ i \neq j. \end{cases} \quad (29)$$

Poacher payoff

$$v(i,j) = \begin{cases} -p_i g_j q_j + (1 - p_i) g_j q_j, & if\ i = j, \\ g_j q_j, & if\ i \neq j. \end{cases} \quad (30)$$

where p_i is probability to meet in forrest i,
q_i , is probability to kill a pray,
g_i is the utility of pray.

Average payoffs

Average payoffs of inspector and poacher

$$U(x, y) = \sum_{i,j} x_i u(i, j) y_j, \tag{31}$$

$$V(x, y) = \sum_{i,j} x_i v(i, j) y_j. \tag{32}$$

Here x_i, y_i are visiting probabilities
of inspector and poacher where i denotes a forrest.

Optimal inspection

Conditions of equal average payoffs

$$\sum_{j=1}^{m} u(i,j)y_j^0 = U,\ i = 1, ..., m$$

$$\sum_{i=1}^{m} v(i,j)x_i^0 = V,\ j = 1, ..., m$$

$$\sum_{i=1}^{m} x_i = 1,\ \sum_{i=1}^{m} y_i = 1 \qquad (33)$$

If there is a feasible solution $x_i \geq 0,\ y_i \geq 0,\ i = 1, ..., m$, that is the equilibrium, if not then additional testing of equilibrium conditions is made, or the additional inequalities are introduced: $x_i \geq \epsilon,\ y_j \geq \epsilon.\ \epsilon > 0.$

Inspector's problem, extended

Inspector's payoff

$$u(i,j) = \begin{cases} p_i g_i q_j, & if\ i = j, \\ 0, & if\ i \neq j, \\ 0, & if\ i = \emptyset. \end{cases} \quad (34)$$

Poacher payoff

$$v(i,j) = \begin{cases} -p_i g_j q_j + (1 - p_i) g_j q_j, & if\ i = j, \\ g_j q_j, & if\ i \neq j, \\ 0, & if\ j = \emptyset. \end{cases} \quad (35)$$

where p_i is probability to meet in forrest i,
q_i is probability to kill a pray,
g_i is the utility of pray,
$\emptyset$ means staying at home.

Average payoffs, extended

Average payoffs of inspector and poacher

$$U(x,y) = \sum_{i,j} x_i u(i,j) y_j, \tag{36}$$

$$V(x,y) = \sum_{i,j} x_i v(i,j) y_j. \tag{37}$$

Here $i, j = 1, 2, ..., m, \emptyset$,
x_i, y_i are probabilities of inspector and poacher actions, where i means going to forrest i or 'staying at home'.
Note that in the extended version, the payoffs do not satisfy the Nash equilibrium conditions, since they are not convex functions of strategies $x_\emptyset, y_\emptyset$, due to jumps at the points $x_\emptyset = 0, y_\emptyset = 0$.

Optimal inspection, extended

Conditions of equal average payoffs

$$\sum_{j=1}^{m,\emptyset} u(i,j)y_j^0 = U,\ i = 1,...,m,\emptyset$$

$$\sum_{i=1}^{m,\emptyset} v(i,j)x_i^0 = V,\ j = 1,...,m,\emptyset$$

$$\sum_{i=1}^{m,\emptyset} x_i = 1,\ \sum_{i=1}^{m,\emptyset} y_i = 1 \qquad (38)$$

If there is a feasible solution $x_i \geq 0,\ y_i \geq 0,\ i = 1,...,m$, that is the equilibrium, if not then additional testing of equilibrium conditions is made, or the additional inequalities are introduced: $x_i \geq \epsilon,\ y_j \geq \epsilon.\ \epsilon > 0.$

Inspection example, simple

If $q_i = g_i = 1$ then from (29)(30)

$$u(i,j) = \begin{cases} p_j, & if\ i = j \\ 0, & otherwise, \end{cases} \tag{39}$$

and

$$v(i,j) = \begin{cases} -p_i + (1 - p_i), & if\ i = j \\ 1, & otherwise. \end{cases} \tag{40}$$

From here

$$p_j y_j = U,\ j = 1, ..., m \tag{41}$$

$$\sum_{i \neq j} x_i + (1 - 2p_j) x_j = V,\ j = 1, ..., m \tag{42}$$

$$\sum_j y_j = 1,\ \sum_i x_i = 1,\ y_j \geq 0,\ x_i \geq 0$$

Two forests

If $m = 2$ then from (41)

$$y_1 = p_2/(p_1 + p_2), \tag{43}$$
$$y_2 = p_1/(p_1 + p_2), \tag{44}$$

and

$$x_1 = p_2/(p_1 + p_2), \tag{45}$$
$$x_2 = p_1/(p_1 + p_2). \tag{46}$$

No ’staying at home’ possibility, in these examples.

$m \geq 2$ forrests

$$u^* = U = p_1 p_2/(p_1 + p_2),$$
$$v^* = V = p_1 + p_2 - 2p_1 p_2/(p_1 + p_2). \quad (47)$$

If $p_1 = 1/3, p_2 = 2/3$

$$x1^* = y_1^* = 2/3,\ x_2^* = y_2* = 1/3,\ u^* = 2/9,\ v^* = 5/9. \quad (48)$$

For any $m \geq 2$

$$y_i = x_i = \prod_{k \neq i} p_k / (\sum_i \prod_{k \neq i} p_k), \quad (49)$$

(50)

where $\prod_{k \neq i} p_k$ is a product of all p_k except p_i.

Duel

Two flying objects are fighting.
The trajectories are

$$dz(t)/dt = az(t), \tag{51}$$

$$dw(\tau)/d\tau = bw(\tau), \tau = 2 - t, \tag{52}$$

Thus

$$z(t) = z_0 e^{at},\ w(\tau) = w_0 e^{b\tau}. \tag{53}$$

Hitting probability

$$p(t) = 1 - d(t)/D. \tag{54}$$

Here $d(t)$ distance between objects,
firing time is t, maximal distance is D.

Payoff functions

Payoff function of the first object:

$$U(t_1, t_2) = \begin{cases} p(t_1) - (1 - p(t_1)), & if \ t_1 < t_2, \\ -p(t_2) + (1 - p(t_2)), & if \ t_2 < t_1, \\ 0, & if \ t_2 = t_1, \end{cases}$$

Payoff function of the second object

$$V(t_1, t_2) = \begin{cases} p(t_2) - (1 - p(t_2)), & if \ t_2 < t_1, \\ -p(t_1) + (1 - p(t_1)), & if \ t_1 < t_2, \\ 0, & if \ t_2 = t_1. \end{cases}$$

One-dimensional duel, "Two Knights"

Payoff function of the first knight:

$$U(t_1, t_2) = \begin{cases} t_1 - (1 - t_1), & if\ t_1 < t_2, \\ -t_2 + (1 - t_2), & if\ t_2 < t_1, \\ 0, & if\ t_2 = t_1, \end{cases}$$

Payoff function of the second knight

$$V(t_1, t_2) = \begin{cases} p(t_2) - (1 - t_2), & if\ t_2 < t_1, \\ -pt_1 + (1 - t_1), & if\ t_1 < t_2, \\ 0, & if\ t_2 = t_1. \end{cases}$$

Equilibrium: $t_1 = t_2 = 0.5$. Here $D = 2$, speed = 1,
Trerefore $p(t_1) = t_1,\ p(t_2) = t_2$.

Optimal duel

Optimal firing time:

$$p(t_1) = p(t_2) = 0.5. \tag{55}$$

Optimal initial heights z_0, w_0
and optimal ascend rates a, b
are calculated by equilibrium conditions
using mixed strategies defined by linear programming.

Economic duel

Dynamical competition of two servers

Profit functions:

$$U_i(T) = \int_{t_0}^{T} u_i(t)dt, \quad (56)$$

$$u_i(t) = a_i(t)y_i(t) - x_i(t), \quad (57)$$

where

$u_i(t)$ is profit of server i, moment t,

$a(t) = \sum_i a_i(t)$ is customer arrival rate.

Service price is $y_i(t)$, service expenses are $x_i(t)$

Trajectories are from:

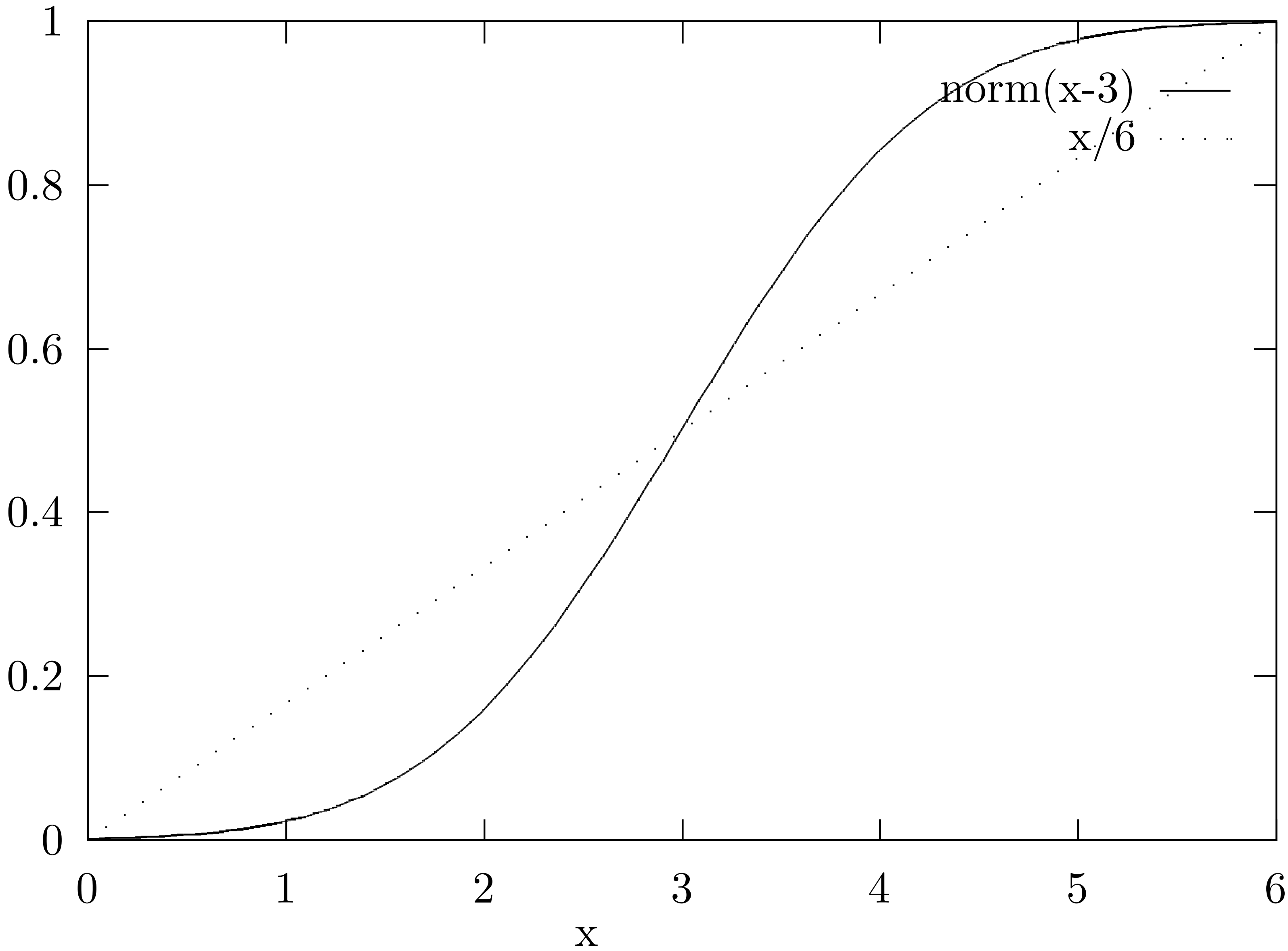
$$dy_i(t)/dt = ay_i(t), \quad (58)$$

$$dx_i(t)/dt = bw_i(t). \quad (59)$$

Optimal economic duel

We search for initial values $y_i(0), x_i(0)$,
and change rates a, b,
that provide equilibrium.
Sever i gets broken at moment t^* if
$U_i(t^*) < -U^*$.
The surviving server enjoys monopolistic profit.

Fig. 2. Utility function



Explaining Fig. 2

Utility function of
- rich person is denoted by dots,
- average person is denoted by continuous line,
- risk area is [0.0, 3.0],
- risk aversion area is [3.0,6.0],
- investment is x,
- total amount is 6.0

Lottery

Utility of event $0 \leq C \leq 1$ is denoted by $u(C)$
and is defined using the lottery:

$$C \sim \{pA + (1-p)B\}. \tag{60}$$

Here utility of keeping C is
$u(C) = p$,
where p is probability to win e $A = 1$,
$1 - p$ is probability to win nothing $B = 0$.
Symbol $\sim$ denotes the "hesitation"
when a player don't know what is better:
- to keep the C,
- or to risk loosing C while trying to win better A.

Expected utility

Expected utility of investment x:

$$U(x) = \sum_{k=1}^{M} u(y^k) p(y^k). \tag{61}$$

Here $y^k = \sum_{i=1}^{m} \delta_i c_i x_i$ is returned wealth,
$u(y^k)$ is utility of wealth y^k,
x_i is capital invested in the object i,
$c_i = 1 + \alpha_i$, α_i is the yield of i,
$\delta_i = 1$ if i survives,
$\delta_i = 0$ if i gets broken,
$p(y^k)$ probability to get wealth y^k.
For example:
$p(y^1) = p_1 \prod_{i \neq 1} (1 - p_i)$.
Here $y^1 = c_1 x_1$, and p_i is survival probability of i.

Optimal insurance

Expected utility of investment x

$$U(x) = \sum_{k=1}^{M} u(y^k)p(y^k).$$

Here $p(y^k)$ is probability to get wealth $y^k = \sum_{i=1}^{m} c_i(x_i)$, and $u(y^k)$ is utility of wealth y^k.

$$c_i(x_i) = \begin{cases} -a_i x_i, & \text{if } \delta_i = 1 \\ -z_i + (1 - a_i)x_i, & \text{if } \delta_i = 0. \end{cases} \tag{62}$$

Explaining insurance variables

In the equation
z_i is price of the object i,
$a_i x_i$ is insurance cost of object i,
$x_i \leq z_i$ is insurance sum of object i.
$\delta_i = 1$ if i survives,
$\delta_i = 0$ otherwise,
$p_i = P\{\delta_i = 1\}$ survival probability of object i.
For example:
$p(y^1) = p_1 \prod_{i \neq 1} (1 - p_i)$,
$y^1 = c_1(x_1) + \sum_{i=2}^{m} c_i(x_i)$, where
$c_1(x_1) = z_1 - a_1 x_1, c_i(x_i) = (1 - a_i) x_i, \ i = 2, ..., m$.

Prediction

ARMA is auto-regression moving-average model

$$w_t = \sum_{i=1}^{p} a_i w_{t-i} + \sum_{i=1}^{q} b_j \epsilon_{t-j} + \epsilon_t. \qquad (63)$$

For example,
w_t is stock rate tomorrow
w_{t-1} is stock rate today,
ϵ_t is a random component tomorrow,

ARMA optimization

a_i, b_j are ARMA parameters defined by minimization of

$$f(x) = \sum_{t=1}^{T} \epsilon_t^2, \qquad (64)$$

where $x = (x_1, ..., x_{p+q}),\ x_i = a_i,\ i = 1, ..., p, x_i = b_{i-p},\ i = p+1, ..., p+q$.

Stock-exchange game

One stock two major customers $i = 1, 2$,
Customer i is buying a stock at moment t if

$$z(t) \leq z_i^{min}(t), \tag{65}$$

where
$z(t)$ stock rate at t,
$z_i^{min}(t)$ is a buying level.
Customer i is selling a stock at t if

$$z(t) \geq z_i^{max}(t), \tag{66}$$

where
$z_i^{max}(t)$ is a selling level.

Stock exchange model

Sock rate at $t+1$ is defined
by the price of previous deal at moment t

$$z(t+1) = \begin{cases} z(t) + \epsilon(t+1), & \text{if } z^{min}(t) < z(t) < z^{max}(t), \\ z^{min}(t) + \epsilon(t+1), & \text{if } z(t) \le z^{min}(t), \\ z^{max}(t) + \epsilon(t+1), & \text{if } z(t) \ge z^{max}(t). \end{cases}$$

Here

$$z^{min}(t) = \max_i \; z_i^{min}(t), \quad (67)$$

$$z^{max}(t) = \min_i \; z_i^{max}(t). \quad (68)$$

Expected profit

Expected profit of player i at next moment $t+1$:

$$\Delta_i(t+1) = \left(\beta_i(t+1) + \delta(t+1) - \alpha(t+1)\right) z(t). \quad (69)$$

Here

$$\beta_i(t+1) = (z_i(t+1) - z(t))/z(t), \quad (70)$$

where $z_i(t+1)$ is expected stock rate at moment $t+1$.

$$\delta(t+1) = d(t+1)/z(t), \quad (71)$$

$d(t+1)$ are expected dividends at moment $t+1$.

$$\alpha(t+1) = a(t+1)/z(t), \quad (72)$$

$a(t+1)$ is yield at moment $t+1$.

Buying and selling levels

Buying level

$$z_i^{min}(t) = k_{buy}\,\Delta_i(t),\; k_{buy} > 1. \tag{73}$$

Selling level

$$z_i^{max}(t) = k_{sell}\,\Delta_i(t),\; k_{sell} < 1. \tag{74}$$

Number of stocks own by customer i at time t

$$N_i(t+1) = \begin{cases} N_i(t), & \text{if } z_i^{min}(t) < z(t) < z_i^{max}(t), \\ N_i(t)+1, & \text{if } z(t) \leq z_i^{min}(t), \\ N_i(t)-1, & \text{if } z(t) \geq z_i^{max}(t) \end{cases}$$

Stock-exchange game, Model 2

One stock two major customers $i = 1, 2$,
Customer i is buying a stock at a moment t if

$$\Delta_i(t) \geq k_i^{buy}, \tag{75}$$

where
$\Delta_i(t)$ is expected profit rate at a moment t, see (80),
Customer i is selling a stock at t if

$$\Delta_i(t) \leq k_i^{sell}, \tag{76}$$

where
$k_i^{buy} > 0$ is a buying level.
$k_i^{sell} < 0$ is a selling level.

Stock-exchange game "Soros"

One additional customer Soros s,
Customer Soros s is buying a stock at a moment t if

$$z(t) \leq z^{buy}(t), \tag{77}$$

Customer Soros s is selling a stock at a moment t if

$$z(t) \geq z^{sell}(t), \tag{78}$$

Here
$z^{buy}(t)$ is a Soros buying price at a moment t.
$z^{sell}(t)$ is a Soros selling price at a moment t. More funds can be made available in this model.

Stock exchange, 2

Sock rate at next moment $t+1$ is defined by the price of previous deal at this moment t

$$z(t+1) = \begin{cases} z(t) + \epsilon(t+1), & \text{if } k_i^{sell} < \Delta_i(t) < k_i^{buy} \text{ for all i}, \\ z^{sell}(t) + \epsilon(t+1), & \text{if } \Delta_i(t) \le k_i^{sell} \text{ for some i}, \\ z^{buy}(t) + \epsilon(t+1), & \text{if } \Delta_j(t) \ge k_j^{buy} \text{ for some j}, \\ z^{sell}(t) + \epsilon(t+1), & \text{if } z(t) \ge z^{sell}(t), \\ z^{buy}(t) + \epsilon(t+1), & \text{if } z(t) \le z^{buy}(t). \end{cases} \tag{79}$$

Expected profit rate, 2

Expected profit rate of player i at a moment $t+1$:

$$\Delta_i(t+1) = (\beta_i(t+1) + \delta(t+1) - \alpha(t+1)). \quad (80)$$

Here

$$\beta_i(t+1) = (z_i(t+1) - z(t))/z(t), \quad (81)$$

where $z_i(t+1)$ is a stock rate at moment $t+1$ predicted by customer i using AR model.

$$\delta(t+1) = d(t+1)/z(t), \quad (82)$$

$d(t+1)$ are expected dividends at moment $t+1$.

$$\alpha(t+1) = a(t+1)/z(t), \quad (83)$$

$a(t+1)$ is yield at moment $t+1$.

Buying and selling stocks, Single level

Number of stocks own by customer i at time t

$$N_i(t+1) = \begin{cases} N_i(t), & \text{if } k_i^{sell} < \Delta_i(t) < k_i^{buy} \text{ for all i}, \\ N_i(t) + N_b, & \text{if } \Delta_i(t) \leq k_i^{sell} \text{ for some i}, \\ N_i(t) - N_s, & \text{if } \Delta_i(t) \geq k_i^{buy} \text{ for some i}. \end{cases} \quad (84)$$

Here N_b is a number of stocks to buy and N_s is a number to sell.

Multi-level example, 2

Sock rate at $t+1$ is defined
by the price of previous deal at a moment t

$$z(t{+}1) = \begin{cases} z(t) + \epsilon(t+1), & \text{if } k_i^{sell}(l) < \Delta_i(t) < k_i^{buy}(l) \text{ for all i,} \\ z^{sell}(t) + \epsilon(t+1), & \text{if } \Delta_i(t) \le k_i^{sell}(l) \text{ for some i,l}, \\ z^{buy}(t) + \epsilon(t+1), & \text{if } \Delta_j(t) \ge k_j^{buy}(l) \text{ for some j,l}, \\ z^{sell}(t) + \epsilon(t+1), & \text{if } z(t) \ge z^{sell}(t), \\ z^{buy}(t) + \epsilon(t+1), & \text{if } z(t) \le z^{buy}(t). \end{cases} \tag{85}$$

Funds and stocks of the customer Soros s are not limited.

Buying and selling stocks, 2

Number of stocks own by customer i at time t in the multi-level mode

$$N_i(t{+}1) = \begin{cases} N_i(t), & \text{if } k_i^{sell}(l) < \Delta_i(t) < k_i^{buy}(l) \text{ for all i}, \\ N_i(t) + N_{i,b}(l), & \text{if } \Delta_i(t) \le k_i^{sell}(l) \text{ for some i}, \\ N_i(t) - N_{i,s}(l), & \text{if } \Delta_i(t) \ge k_i^{buy}(l) \text{ for some i}. \end{cases} \tag{86}$$

Here
$k_i^{buy}(l+1) > k_i^{buy}(l)$, $k_i^{sell}(l+1) < k_i^{buy}(l)$,
$N_{i,b}(l+1) < N_{i,b}(l),\ N_{i,s}(l+1) < N_{i,s}(l),$
$N_i(t+1) = N_i(t),\ if\ \Delta_i(t) \le k_i^{sell}(l)\ and\ no\ funds\ left\ for\ i$.

Number of stocks for buying and selling, 2

Maximal number of stocks to buy at the time t

$$N_{i,b}(t) = (C_i(t) + U_i(t, T_0))/z(t), \tag{87}$$

Here $C_i(t)$ is credit available for customer i at time t,
$U_i(t, T_0)$ is customers' i profit accumulated at time t.
Number of stocks to buy at the time t and buying level l

$$N_{i,b}(t, l) = N_{i,b}/l, \tag{88}$$

Maximal number of stocks to sell at the time t

$$N_{i,s}(t) = N_i(t), \tag{89}$$

Number of stocks to buy at the time t and buying level l

$$N_{i,s}(t, l) = N_{i,s}/l, \tag{90}$$

$l = 1, 2, 3$ as usuall.

Customer profits, 2

Profit of customer i during time $T_0 \leq t \leq T$:

$$U_i(T, T_0) = \tag{91}$$

$$N_i(T) z_T - N(T_0) z_{T_0} - \sum_{t=T_0}^{T} \left(N_i(t+1) - N_i(t)\right) z(t).$$

Profit u_i depends on accuracy of prediction $\beta_i(t+1)$ and random events $\epsilon(t)$.
If customer i predicts by AR model then

$$z_i(t+1) = \sum_{k=1}^{p_i} a_i^k \; z_{t-k} + \epsilon_i(t+1). \tag{92}$$

Parameters of customer predictions

AR parameters a_k are defined by condition:

$$\min_{a_i} \sum_{s=t_0}^{t} \epsilon_i^2(s), \tag{93}$$

where

$$\epsilon_i(s) = z(s) - \sum_{k=1}^{p_i} a_i^k \; z(s-k). \tag{94}$$

This model is for stock exchange simulation
by generating time series of virtual stock rates.
The model is not intended for actual predictions.

Stock exchange equilibrium

We search for AR parameters (p_1, p_2) satisfying Nash equilibrium
using mixed strategies

$$x_{p_i},\ i=1,2,\ p_i=1,...,P_i,\ \sum_{p_i=1}^{P_i} x_{p_i}=1, \qquad (95)$$
$$0 \le x_{p_i} \le 1.$$

Here x_{p_i} is a probability of parameter p_i.
Denote the "no-contract" vector

$$x_{p_1}^1 = \arg\max_{x_{p_1}} U_1^K(T_0, T, x_{p_1}, x_{p_2}^0), \qquad (96)$$

$$x_{p_2}^1 = \arg\max_{x_{p_2}} U_1^K(T_0, T, x_{p_1}^0, x_{p_2}) \qquad (97)$$

where $x_{p_i}^0,\ i=1,2$ is a "contract" vector.

Search for Nash equilibrium

Nash equilibrium:

$$(x^*_{p_1}, x^*_{p_2}) = \arg \min_{x^0_{p_1}, x^0_{p_2},} ||(x^1_{p_1}, x^1_{p_2}) - (x^0_{p_1}, x^0_{p_2})||^2 \quad (98)$$

Optimizing by global stochastic methods.
If minimum not small enough
testing equilibrium conditions
by analysis of profit function convexity.
Final objective
is to define optimal AR parameters.
The results are unexpected- equilibrium is by Wiener model
that means "no prediction".

Sharpe Ratio

Sharpe ratio is defined as:

$$S = \frac{E[R_a - R_b]}{\sigma} = \frac{E[R_a - R_b]}{\sqrt{\mathrm{var}[R_a - R_b]}}, \tag{99}$$

where R_a is the asset return, R_b is the return on a benchmark asset, $E[R_a - R_b]$ is the expected value of the excess of the asset return over the benchmark return, and σ is the standard deviation of this expected excess return. Expected return of portfolio of assets with weights:

$$\mathrm{E}(R_p) = \sum_i w_i \, \mathrm{E}(R_i) \tag{100}$$

where R_p is the return on the portfolio p, R_i is the return on asset i, $w_i \geq 0$ is the weighting of component asset i (that is, the share of asset i in the portfolio), and $\sum_i w_i = 1$.

Sharpe Ratio-2

Using these symbols, the portfolio return variance: can be written as:

$$\sigma_p^2 = \sum_i \sum_j w_i w_j \operatorname{cov}(R_i R_j), \tag{101}$$

. Portfolio return volatility (standard deviation):

$$\sigma_p = \sqrt{\sigma_p^2} \tag{102}$$

Call center model

Calls are random generated by distribution
$F_a(t) = P\{\tau < t\}$.
defining probability that time τ
until next call is less than t.
Parameter a denotes call rate- average number of calls by a time unit.

Generating arrival times

In the Poisson stream the arrival time
of next customer is defined by condition

$$F_a(t) = 1 - e^{-1/at}, \tag{103}$$

$$\tau = -aln(1 - \xi). \tag{104}$$

where ξ random number uniform in [0,1].
If service time is exponential

$$F_x(t) = 1 - e^{-1/mxt}, \tag{105}$$

$$\tau = -xln(1 - \xi), \tag{106}$$

where mx is service rate of m agents,
then expected waiting time in Poisson stream is

$$\gamma = \frac{a}{mx(mx - a)}, \tag{107}$$

Optimizing number of agents

If x is service rate of single agent,
and m is number of agents
then total service expenses are

$$c(a,m) = c_m m + c_\gamma(a)\gamma \tag{108}$$

where c_m expenses for one agent,
$c_\gamma(a)$ estimated cost of waiting time.
Optimal number of agents

$$m(a) = \min_m c_(a,m). \tag{109}$$

Optimizing m important is estimated rate a.

Scale model

Predicting call rate

$$Z_i = z_{p(i)} \; s_{p(i)}. \tag{110}$$

where
$Z_i = (z_{ij},\ j = 1, ..., m)$ predicted graph for day i,
z_{ij} hour j,
$z_{p(i)}$ average of day $p(i)$,
$p(i)$ defines a previous day similar to the day i.

Similarity conditions

For example,
similar are days such as Sunday, Saturday, working day.
suPanašūs būna sekmadieniai, šeštadieniai.
Thus assumption that the hourly graph of next Sunday will be similar to previous Sunday,
the scale is defined by

$$s_{p(i)} = \frac{z_{i-1}}{z_{p(i-1)}}. \tag{111}$$

Here z_{i-1} ǎverage of this day,
$z_{p(i-1)}$ average of similar previous day.

Optimal scheduling

Flow-shop problem

Here the sequence of tools is defined by technology
for example in tailors shop:
scissors,needle, and iron.
Operation times τ_{ij} define time
to perform a task i by a tool j
For example, in tailor shop τ_{ij} is
a time to cut a suit i using scissors j.
Objective is make-span minimization
In small scale flow-shop problems
we can optimize by comparing all task sequences,
for example: $2, 1, 4, 6, 5, 3, 7$.
Optimizing large scale flow-shop problems
heuristic methods are applied, as usual.
Convergence can be provided by randomization.

School scheduling

Here tasks represent academic classes. Tools are topics.
School resources are teachers and class-rooms.
Sequence of tools is free.
Objective is minimization of "penalty points". Penalty points define deviation from the "perfect schedule".
In the web-site examples the initial school schedule is improved by permutations. Optimization is made by Simulated Annealing with parameters optimized using Bayesian approach.

Sequential decisions

Dynamic programming

"Single bride" problem

No divorce. Number of proposals is N.
Goodness of grooms ω is defined by Gaussian distribution:

$$p(\omega) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-1/2(\frac{\omega-\alpha_0}{\sigma_0})^2}, \tag{112}$$

where α_0 is average goodness ,
σ_0^2 is goodness variance.
Brides impression is Gaussian, too:

$$p(s|\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2(\frac{s-\omega}{\sigma})^2}. \tag{113}$$

here σ error of brides judgment.

Posterior goodness

Posterior goodness of grooms regarding impression s

$$p(\omega|s) = \frac{p(s|\omega)p(\omega)}{\int_{-\infty}^{\infty} p(s|\omega)p(\omega)d\omega}. \tag{114}$$

Expected goodness of grooms making impression s

$$u(s) = \int_{-\infty}^{\infty} \omega p(\omega|s)d\omega, \tag{115}$$

where $p(\omega)$ is density of probability. If ω and s discrete

$$u(s_j) = \sum_i \omega_i P(\omega_i|s_j), \tag{116}$$

where $P(\omega_i|s_j)$ is probability of goodness given $s = s_j$.

Example of posterior probability

Prior probability of rain $p(r) = 0.5$, that of clear $p(c) = 0.5$.
Observed probabilities of wrong/right predictions:
$p(s = c|r) = 0.1$, $p(s = r|r) = 0.9$,
$p(s = r|c) = 0.2$, $p(s = c|c) = 0.8$.
The posterior probability of wrong rain prediction

$$p(r|s = c) = \\ p(s = c|r)p(r)/(p(s = c|r)p(r) + p(s = c|c)p(c)) = \\ 0.1 * 0.5/(0.1 * 0.5 + 0.8 * 0.5) = 0.109. \tag{117}$$

The posterior probability of wrong clear prediction

$$p(c|s = r) = \\ p(s = r|c)p(c)/(p(s = r|c)p(c) + p(s = r|r)p(r)) = \\ 0.2 * 0.5/(0.2 * 0.5 + 0.9 * 0.5) = 0.182. \tag{118}$$

Bride's decision function

Expected goodness of the last proposal N

$$u_N(s) = u(s), \tag{119}$$

because bride must marry at least once
by problem definition.
Optimal decision $d_{N-1}(s)$ for ($N-1$)-th proposal:

$$u_{N-1}(s) = \max_d(du(s) + (1-d)u_N), \tag{120}$$

$$d_{N-1}(s) = arg\max_d(du(s) + (1-d)u_N). \tag{121}$$

Optimal decision for ($N-n$)-th proposal
is defined in a similarly.
This way bride's decision function is build.
This function shows how the optimal decision
depends on the impression s made by the proposal n.

Simple example-1

Probability density p of goodness ω

$$p(\omega) = \begin{cases} 0.5, & \text{if } -1 < \omega < 1, \\ 0, & \text{othervise} \end{cases} \tag{122}$$

Brides' observation $s = \omega$ Then expected goodness of the last proposal N

$$u_N(s) = u(s) = s, \tag{123}$$

Optimal decision $d_{N-1}(s)$ for ($N-1$)-th proposal:

$$u_{N-1}(s) = \max_d (d * s + (1-d) * 0), \tag{124}$$

$$d_{N-1}(s) = arg \max_d (d * s + (1-d) * 0). \tag{125}$$

Simple example-2

From here the optimal decission

$$d^*_{N-1}(s) = \begin{cases} 1, & \text{if } s > 0, \\ 0, & \text{if } s < 0 \end{cases} \quad (126)$$

and the optimal goodness
$u^*_{N-1} = \max(0, s)$
the expected optimal goodness at $N - 1$

$$Eu^*_{N-1} = \int_{-1}^{1} \max(0, s) p(s) ds = \quad (127)$$

$$0.5 \int_{0}^{1} s ds = 0.25 \quad (128)$$

Simple example-3

Optimal decision for $(N-2)$-th proposal

$$u_{N-2}(s) = \max_d (d * s + (1-d) * 0.25), \tag{129}$$

$$d_{N-2}(s) = arg \max_d (d * s + (1-d) * 0.25). \tag{130}$$

From here

$$d_{N-1}(s = \begin{cases} 1, & \text{if } s > 0.25, \\ 0, & \text{if } s < 0.25 \end{cases} \tag{131}$$

The remaining steps are defined similarly.
This way bride's decision function is build.
This function shows how the optimal decision depends on the impression s made by the current proposal.

"Free" bride (Buy-a-PC)

Decision function

Bride is free to marry and to divorce N times.
Denote groom's (new PC) goodness by ω.
Assume "clairvoyant" bride
that defines goodness $\omega = s$ exactly.
Goodness of actual husband (old PC) denote by q.
Expected "goodness" of the N-th proposal:

$$u_N(\omega, q_N) = \max_d (d\omega + (1-d)q_N). \quad (132)$$

Here the optimal decision depends on both components: goodness of husband (keeping the old PC) q and goodness of proposal (new PC) ω:

$$d_N^* = \begin{cases} 1, & \text{if } \omega_N > q_N - c_N, \\ 0, & \text{if } \omega_N \le q_N - c_N \end{cases} \quad (133)$$

Utility goodness

N-th proposal:

The utility of the decision $d = 0$, to keep the old PC in the last year N , is $q_N - c_N$.
Here q_N is the utility of old PC, $c_N = \tau_N - g_0(N)$ is the penalty of refusing to buy a new PC. This includes the waiting losses τ_N minus the price $g_0(N)$ of new PC in the year N.
It is assumed that we abandon the old PC as soon as we obtain the new one. Therefore, one "wins" the price $g_0(N)$ of the new PC by using the old PC.
Bride is free to marry and to divorce N times.

Utility vector- linear approximation

N-th proposal: Define PC parameters by a vector $g = (g_0, g_1, g_2, g_3)$. Here g_0 is the market price of PC in $, g_1 is the speed of CPU in MHz, g_2 is the volume of RAM in MB, and g_3 is the volume of HD in GB.
Express a subjective utility of PC by the weighted sum

$$\omega = a_1 g_1 + a_2 g_2 + a_3 g_3. \tag{134}$$

Here a_i is user evaluations of utility of quality parameter g_i expressed in $ per unit. The users evaluation ω differs from the market price g_0, as usuall.
The PC utility defined as a sum of utilities of different components is just first approximation. Therefore, we need to evaluate different PC configurations separately.

Utility function

N-th proposal:

The problem is how to define user evaluation based on the utility theory.

Step 1: define a set of events E as a set of PC described by different feasible vectors (g_1, g_2, g_3, g_4)

Step 2: define a sequence of events $E_i, i = 1, ..., I$ ordered by the condition $E_{i-1} \leq E_i \leq E_{i+1}$. Condition $E_I \leq E_{i+1}$ means that we prefer PC E_{i+1} to E_i.

Step 3: set the normalized utility functions $u_0(E_1) = 0, u_0(E_I) = 1$

Step 4: define the remaining utility functions $u_0(E_i) = p_i$ where p_i is the 'hesitation' probability determined by the lottery:

$$E_i \sim \{p_i E_I + (1 - p_i) E_1\}. \tag{135}$$

Relation of quality and price

N-th proposal: Define by h rhe highest price a user is ready to paye for the best PC. Then the general utility $u(E_i)$ of the PC of lesser configuration E_i is

$$u(E_i) = h * u_0(E_i) - g_{0i} \tag{136}$$

Or, in case of linear approximation:

$$u(E_i) = \omega_i - g_{0i} \tag{137}$$

where ω_i is goodness of PC E_i calculated by (134) and g_{0i} is the market price.

Maximization of expected goodness

Regarding ($N-i$)-th proposal

Optimal goodness depends on q and ω.

$$u_{N-i}(\omega, q) = \max_d (d\omega + (1-d)(u_{N-i+1}(q_{N-i+1}) - c_{N-i})).$$

Here $u_{N-i+1}(q)$ is expected goodness of ($N-i+1$)-th proposal given husband goodness q.

$$u_{N-i+1}(q) = \int_{-\infty}^{\infty} u_{N-i+1}(\omega, q) p_{N-i+1}(\omega) d\omega. \tag{138}$$

Decision function

Regarding ($N-i$)-th proposal

$p_{N-i+1}(\omega)$ is probability of goodness ω of ($N-i+1$)-th proposal.

$$q_{N-i+1} = \begin{cases} \omega_{N-i}, & \text{if } q_{N-i+1} < q^*_{N-i}, \\ q_{N-i}, & \text{if } q_{N-i+1} \geq q^*_{N-i} \end{cases}$$

Here q^*_{N-i} is defined by

$$\omega_{N-i} = u_{N-i+1}(q^*_{N-i}) - c_{N-i}. \tag{139}$$

Optimal decision regarding ($N-i$)-th proposal

$$d^*_{N-i} = \begin{cases} 1, & \text{if } \omega_{N-i} > u_{N-i+1}(q_N - i + 1) - c_{N-i}, \\ 0, & \text{if } \omega_{N-i} \leq u_{N-i+1}(q_{N-i+1}) - c_{N-i} \end{cases}$$

Here decision function d^*_{N-i} depends on both ω and q.

Diet problem

$$min_x \sum_{i=1}^{m}((c_i - s_i)x_i + g(\sum_{i=1}^{m} a_{i1}x_i - b_1)), \quad (140)$$

$$\sum_{i=1}^{m} a_{ij}x_i \geq b'_j, \ \sum_{i=1}^{m} a_{ij}x_i \leq b''_j, \ j = 1, ..., n \quad (141)$$

$$x_i \geq 0, \ i = 1, ..., m. \quad (142)$$

For example,
c_1 is a price of bread, a_{11} are calories of bread, b'_1 are necessary calories, b''_1 are harmful calories, s_i is the taste (expressed in money units) , g beauty factor (expressed in money units).
Second inequality (141) is not included into LP, it is a warning.

Diverse diet problem

$$min_x \sum_{i=1}^{m}((c_i - s_i)x_i + g(\sum_{i=1}^{m} a_{i1}x_i - b_1) - d(\sum_{i=1}^{m} d_i x_i)), \quad (143)$$

$$\sum_{i=1}^{m} a_{ij}x_i \geq b'_j, \ \sum_{i=1}^{m} a_{ij}x_i \leq b''_j, \ j = 1, ..., n \quad (144)$$

$$x_i \geq 0, \ i = 1, ..., m. \quad (145)$$

Here
d is the taste of food diversity (expressed in money units),
d_i is the dish indicator: $d_i = 1$, if x_i is a dish, $d_i = 0$, otherwise,
If x_i is a dish then $x_i = int$ and $0 \leq x_i \leq 1$.

Longer term diet problem

$$min_x \sum_{i=1}^{m}((c_i - s_i)x_i + g(\sum_{i=1}^{m} a_{i1}x_i - Tb_1) - d(\sum_{i=1}^{m} d_i x_i)) \quad (146)$$

$$\sum_{i=1}^{m} a_{ij}x_i \geq Tb'_j, \ \sum_{i=1}^{m} a_{ij}x_i \leq Tb''_j, \ j = 1, ..., n \quad (147)$$

$$\sum_{i=1}^{m} d_i x_i \leq T, \ \sum_{i=1}^{m} d_i \geq T. \quad (148)$$

Here d is the taste of food diversity, T is the time period, d_i is the dish indicator: $d_i = 1$, if x_i is a dish, othervise $d_i = 0$. If x_i is a dish then $x_i = int$ and $0 \leq x_i \leq 1$.

Combinatorial diet problem

$$min_x(\sum_{i=1}^{m}(c_i - s_i)x_i + g(\sum_{i=1}^{m} a_{i1}x_i - Tb_1) - d(x) + b\sum_{j=1}^{n}(B'_{j+} + B''_{j+})), \quad (149)$$

Here

$$B'_j = (-\sum_{i=1}^{m} a_{ij}x_i + Tb'_j),\ B''_j = (\sum_{i=1}^{m} a_{ij}x_i - Tb''_j), \quad (150)$$

B'_{j+}, B''_{j+} are positive parts of B'_j, B''_j, $d(x)$ is a function defining the taste of food diversity (for example, $d(x)$ could be a number of different non-zero components x_i in the diet x), b is a penalty factor for violation constraints. (147)

Solving combinatorial diet problem

Step 1. Set an initial diet $x = x^1$ and evaluate its quality by calculating the sum:

$$D(x^1) = (\sum_{i=1}^{m}(c_i - s_i)x_i^1 + g(\sum_{i=1}^{m} a_{i1}x_i^1 - Tb_1) - d(x^1) + b\sum_{j=1}^{n}(B'_{j+} + B''_{j+})) \quad (151)$$

Step 2. Generate next diet x^2 by changing randomly some dishes and evaluate the quality $D(x^2)$.
Step 3. If $h_2 = D(x^2) - D(x^1) \leq 0$ go to x^2.
If $h_2 > 0$ go to x^2 with probability r_2;
keep x^1 and return to step 1 with probability $1 - r_2$.

Combinatorial diet problem-2

Probability r_2 is generated by SA formula 231 defined in the slide "Simulated Annealing"
Parameter x of SA is optimized using GMJ system.
Bayesian Heuristic Approach (BHA) is recommended method for improving a user defined initial diet by optimizing parameters of Simulated Annealing (SA).

Defining SA parameters

Step 1. Generate probability r_2 by SA formula 173.
Step 2. Optimize parameter x of SA by GMJ.
The standart reference to GMJ is this:

```
public Domain domain ()
return domain;
public double f (Point pt)
..............
return f
```

Here 'domain' defines constraints of SA variables,
'pt' is vector of SA variables,
'f' is the 'goodness' function of the best diet
after 'IT' iterations using fixed SA variables.

Simplex algorithm

Simple example:

$$min_x z \tag{152}$$

$$z = x_1 - x_3 \tag{153}$$

$$x_1 + x_2 + x_3 = 1, x_i \geq 0 \tag{154}$$

Here x_2 base variable,
$x_1,\ x_3$ free variables. $x_1 = 1$ base solution obtained when free variables are equal to zero
$x_1 = x_3 = 0$, then $z = 0$.
This base solution is improved by increasing x_3, because $c_3 = -1 < 0$.
New base solution $x_3 = 1, x_1 = x_2 = 0$, where $z = -1$, can't be improved,
since both free variables are non-negative $c_1 = 1, c_2 = 0$.

Knapsack problem

Integer Linear Programming

$$\max_x \sum_{i=1}^{m} c_i x_i, \tag{155}$$

$$\sum_{i=1}^{m} g_i x_i \leq g, \tag{156}$$

$$x_i = \{0, 1\}. \tag{157}$$

Here c_i is price of the object i and g_i is weight of object i,
g is weight limit.
$x_i = 1$ means to take object i, $x_i = 0$ means to leave it.
For small scale problems Branch-and-Bounds are used,
for large scale heuristics are applied, as usual.

Method of Branch-and-Bounds

Example is the knapsack problem:

$$\max_x \sum_{i=1}^{m} c_i x_i \tag{158}$$

$$\sum_{i=1}^{m} g_i x_i \leq g, \ x_i = \{0, 1\}. \tag{159}$$

Calculating Bounds

Bounds are obtained by the auxiliary LP problems:

$$C_1 = c_1 + \max_x \sum_{i=2}^{m} c_i x_i \tag{160}$$

$$g_1 + \sum_{i=2}^{m} g_i x_i \leq g \tag{161}$$

$$C_0 = \max_x \sum_{i=2}^{m} c_i x_i \tag{162}$$

$$\sum_{i=2}^{m} g_i x_i \leq g \tag{163}$$

Branch 1- take the object, branch 0- leave the object.
If $C_0 < c_1$- cut branch 0, if $C_0 \geq c_1$- keep branching.

Worst case

Worst case is when
$c_i = c,\ g_i = g_i,\ i = 1, ..., m$.
Here no branch is cut and all $N = 2^m$ knapsacks are regarded.
Difficult case is when
$h_i = c_i/g_i = h,\ i = 1, ..., m$.
Here just a few branches are cut.
Favorable case is when h_i differ.
Here many branches are cut.

Heuristic methods

$$\max_x \sum_{i=1}^{m} c_i x_i, \tag{164}$$

$$\sum_{i=1}^{m} g_i x_i \leq g, \;\; x_i = \{0, 1\}. \tag{165}$$

Here heuristics

$$h_i = c_i / g_i. \tag{166}$$

Greedy heuristics is to take the best object

$$i^* = \arg\max_i h_i. \tag{167}$$

Here $N \leq m^2$ if h_i differ. If $h_i = const$- the greedy heuristic don't work, randomization is applied.

Mixing heuristics

Example is the knapsack problem.
Randomized heuristic means taking object i with probability r_i.
Traditional randomization

$$r_i = h_i / \sum_j h_j. \tag{168}$$

works well if h_i differ. If all h_i are equal this means uniform random search.
Example of mixed randomization:

$$r_i^0 = 1/m, \tag{169}$$

$$r_i^1 = h_i / \sum_j h_j, \tag{170}$$

$$r_i^\infty = 1,\ jei\ h_i = \max_j h_j. \tag{171}$$

Optimizing mixture of heuristics

Example is a lottery of three heuristics: $x = (x_0, x_1, x_2)$. Here
x_0 is a probability to "win" the greedy heuristic,
x_1 is a probability to "win" the randomized greedy heuristic,
x_2 is a probability to "win" the Monte Carlo search.
Lottery x is optimized using Bayesian algorithm searching for x providing best average results after K iterations.
That is a simple example of Bayesian Heuristic Approach (BHA).
Another example of BHA is school scheduling where parameters of Simulated Annealing (SA) are optimized.
Third example of BHA is flow-shop problem where mixture of three heuristics is optimized.

Optimization complexity

Algorithm is polynomial if the computing time $T = Cm^K$.
Here K is an integer and m is complexity.
In discrete problems m is number of variables.
In continuous problems m is accuracy,
meaning that error $\epsilon \leq 2^{-m}$.
Algorithm is exponential, if the computing time $T \geq C2^m$.
Important are NP-complete problems.
Simplest examples are knapsack and traveling salesman problems.
No polynomial algorithm is known for NP-complete problems.
However there is no proof that exponential algorithm is needed.

Complexity examples

Linear programming is polynomial:
simplex algorithm is exponential but interior point is polynomial.
Knapsack, flow-shop and traveling salesmen problems all are NP-complete.
Global optimization of continuous functions is exponential, in general.
Here computing time
$T \geq C2^{mn}$,
where
m is accuracy,
n is number of variables.

Local optimization

Descent methods

Objective is minimization of function
$f(x), \;\; x = (x_i, \; i = 1, ..., m)$,

$$x^{n+1} = x^n - \alpha_n s_n. \qquad (172)$$

Here α is step size and s_n is step direction.

$$\alpha_n = \arg min_\alpha f(x^n - \alpha_n s_n), \qquad (173)$$

where n is iteration number.
Simple case is gradient algorithm when

$$s_n = grad \; f(x^n). \qquad (174)$$

Here the gradient is

$$grad \; f(x^n) = (\frac{\partial f(x^n)}{\partial x_i}, \; i = 1, ..., m). \qquad (175)$$

Newton's method

In Newton's method the step direction is

$$s_n = H_n^{-1} grad\ f(x^n), \tag{176}$$

where the Hessian

$$H_n = (\frac{\partial^2 f(x^n)}{\partial x_i\ \partial x_j},\ i, j = 1, ..., m). \tag{177}$$

Quasi-Newton's method

In Quasi-Newton's (variable Metrics) method step direction is

$$H_n^{-1} \approx G_n, \tag{178}$$

where

$$G_{n+1} = G_n - \frac{(G_n\gamma_n)(G_n\gamma_n)^T}{\gamma_n^T G_n \gamma_n} + \frac{\delta_n \; \delta_n^T}{\delta_n^T \; \gamma_n}. \tag{179}$$

Here symbol T denotes transposition,

$$\gamma_n = grad\ f(x_n) - grad\ f(x_{n-1}), \tag{180}$$

$$\delta_n = x^n - x^{n-1}, \tag{181}$$

$$G_0 = I. \tag{182}$$

Local convergence

Gradient method converges,
if $f(x)$ is a convex differentiable function.
Newton's and Quasi-Newton's methods converge, if $f(x)$ is convex twice-differentiable function.
Gradient method converges slowly:

$$||x^n - x^*|| \to 0. \quad (183)$$

Newton's and Quasi-Newton's methods converge fast:

$$\frac{||x^n - x^*||}{||x^{n-1} - x^*||} \to 0, \quad (184)$$

where x^* is the optimum.
The main difficulty of Newton's method is calculation of inverse Hessian H_n^{-1}. Newton's method fails if $\det H_n = 0$.
No inverse Hessian is needed using Quasi-Newton's.

Constrained optimization

$$\max_x \; f_0(x), \tag{185}$$

$$f_j(x) \leq c_j, \; j = 1, ..., n. \tag{186}$$

Penalty function:

$$\max_x \{f_0(x) - \sum_j b_j (f_j(x) - c_j)^2\}, \tag{187}$$

$$b_j = \begin{cases} b, & \text{jei } f_j(x) > c_j, \\ 0, & \text{jei } f_j(x) \leq c_j \end{cases} \tag{188}$$

It is difficult to define right penalty factor b: if b is small then constraits are violated, if b is great then we optimize the penalties instead of the original function $f(x)$.

Lagrange multipliers

Constrained optimization

$$\max_x \; f_0(x), \tag{189}$$

$$f_j(x) \le c_j, \; j = 1, ..., n. \tag{190}$$

Optimization of Lagrange function

$$\min_{\lambda > 0} \max_x \{f_0(x) - \sum_j \lambda_j \; (f_j(x) - c_j)\}. \tag{191}$$

Economics of Lagrange multipliers

$$\min_{\lambda>0} \max_{x} \{f_0(x) - \sum_j \lambda_j \ (f_j(x) - c_j)\}. \quad (192)$$

Economic interpretation of Lagrange function:
$f(x)$ is profit of factory owner, c_j is available resource j,
x is technology chosen by factory owner,
λ_j is the price for additional resource j set by owner of the resource,
$\min_{\lambda>0}$ maximizes profit of resource owner,
$\max_x$ maximizes profit of factory owner at fixed prices λ for additional resources.
If all $f_j(x), j = o, .., n$ are convex then minimization of Lagrangian provides minimum of original constrained optimization problem.

Stochastic Gradient

Minimize

$$\min_x \; f(x). \tag{193}$$

$$\tag{194}$$

When we observe the sum:

$$\phi(x) = f(x) + \xi \tag{195}$$

where ξ is Gaussian noise $(0, \sigma)$.
Then the stochastic gradient:

$$x^{n+1} = x^n - a_n/s_n \left(\psi(x^n + s_n) - \psi(x^n)\right). \tag{196}$$

Stochastic Gradient, Convergence

The sequence 196 converges with probability 1 to the minimum of convex differentiable fumction $f(x)$ if:

$$\lim_{n\to\infty} s_n = 0, \tag{197}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i/s_i = \infty, \tag{198}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (a_i/s_i)^2 = C < \infty. \tag{199}$$

where n is iteration number.
For example,
$s_n = 1/n,\ a_n = 1/n^2$

Stochastic Linear Programming 1

$$x + 2y = c \tag{200}$$

$$x + y = a \tag{201}$$

where c is cost, x is production, $a \geq 0$ is uncertain demand, y is amount to buy.
Expected cost:

$$f(x) = xP(a \leq x) + 2(a - x)P(a \geq x) \tag{202}$$

where $P(a \leq x)$ is probability that demand will not exceed production.
Suppose that probability density

$$p(a) = \begin{cases} 1, & \text{if } 0 \leq a \leq 1, \\ 0, & \text{otherwise} \end{cases} \tag{203}$$

Stochastic Linear Programming 2

From here:

$$f(x) = \int_0^x x da + \int_x^1 2(a-x) da = \tag{204}$$

$$x^2 + 1 - x^2 - 2x(1-x) = 1 - 2x + 2x^2 \tag{205}$$

minimum $f(x) = 1/2$ is reached when $x = 1/2$
and is equal to expected demand:
$Ea = \int_0^1 a da = \ 1/2$
but not always, they say

Global continuous optimization

Uniform grid

$$r_n = \max_x \min_i ||x - x^i||. \tag{206}$$

Grids $x(n) = (x^i,\ i = 1, ..., n)$ that minimize $r_n^0 = \min_{x^i, i=1,...,n} r_n$, provide best accuracy for Lipschitz functions

$$\frac{||f(x^i) - f(x_j)||}{||x^i - x_j||} \leq L < \infty, \tag{207}$$

where L is Lipschitz constant. Here best accuracy means minimization of maximal deviation ϵ from the global optimum

$$\epsilon = LR_n, \tag{208}$$

$$R_n = arg \min_{x(n)} \ r_n. \tag{209}$$

Pareto-Lipschitzian optimality (PO)

Objective is to minimize Lipschitz-function $f_L(x)$ with unknown Lipschitz constant L.
The decision x dominates the decision x^* if

$$f_L(x) \leq f_L(x^*), \ for \ all \ L \tag{210}$$

$$f_L(x) > f_L(x^*), \ for \ at \ least \ one \ L \tag{211}$$

The decision x^* is Pareto Optimal (PO) if there is no dominant x.

Pareto-Lipschitzian optimization (PLO)

The variables x are represented by the intervals $i : a_i \leq x \leq b_i$ and the function $f_L(x)$ is approximated by the lower bounds $f(x)$

$$f_L(x) \leq f(c_i) - L * l_i/2, \; a_i \leq x \leq b_i. \tag{212}$$

Here $c_i = (b_i + a_i)/2, \; l_i = b_i - a_i$. The interval $i : a_i \leq x \leq b_i$ dominates the interval j, if

$$f(c_i) - L * l_i/2 \leq f(c_j) - L * l_j/2, \; for \; all \; L \tag{213}$$

$$f(c_i) - L * l_i/2 < f(c_j) - L * l_j/2, \; for \; at \; least \; one \; L \tag{214}$$

The interval j is Pareto Optimal (PO) if there is no dominant interval i.

Examples of (PLO)

$Example\ 1$

The lenghts:

$l_1 = 3,\ l_2 = 2,\ l_3 = 1.$

The function values:

$f(c_1) = 3,\ f(c_2) = 2,\ f(c_3) = 2$.

The intervals $1\ and\ 3$ are PO, the interval 2 is not PO.

$Example\ 2$

The lenghts:

$l_1 = 3,\ l_2 = 2,\ l_3 = 1.$

The function values:

$f(c_1) = 3,\ f(c_2) = 2,\ f(c_3) = 1$.

Here all the intervals $1\ and\ 2\ and\ 3$ are PO.

Defining user preferences

A convenient way to represent user preferences is by supplying an importance measure to each multi-criteria component L. Since the set of the Lipschitz functions are continuous, the proper measure would be the probability density $p(L)$. Then the PO interval

$$i(p) = arg\ min_i \int_L (f(c_i) - L * l_i/2)p(L)dL = \\ arg\ min_i(f(c_i) - E\{L\} * l_i/2). \tag{215}$$

were $E\{L\}$ is the expected value of the Lipschitz constant L.

If, for example $p(L) = exp(-L)$ then $E\{L\} = 1$. In standard applications the criteria set is discrete, so, the integral is replaced by sum.

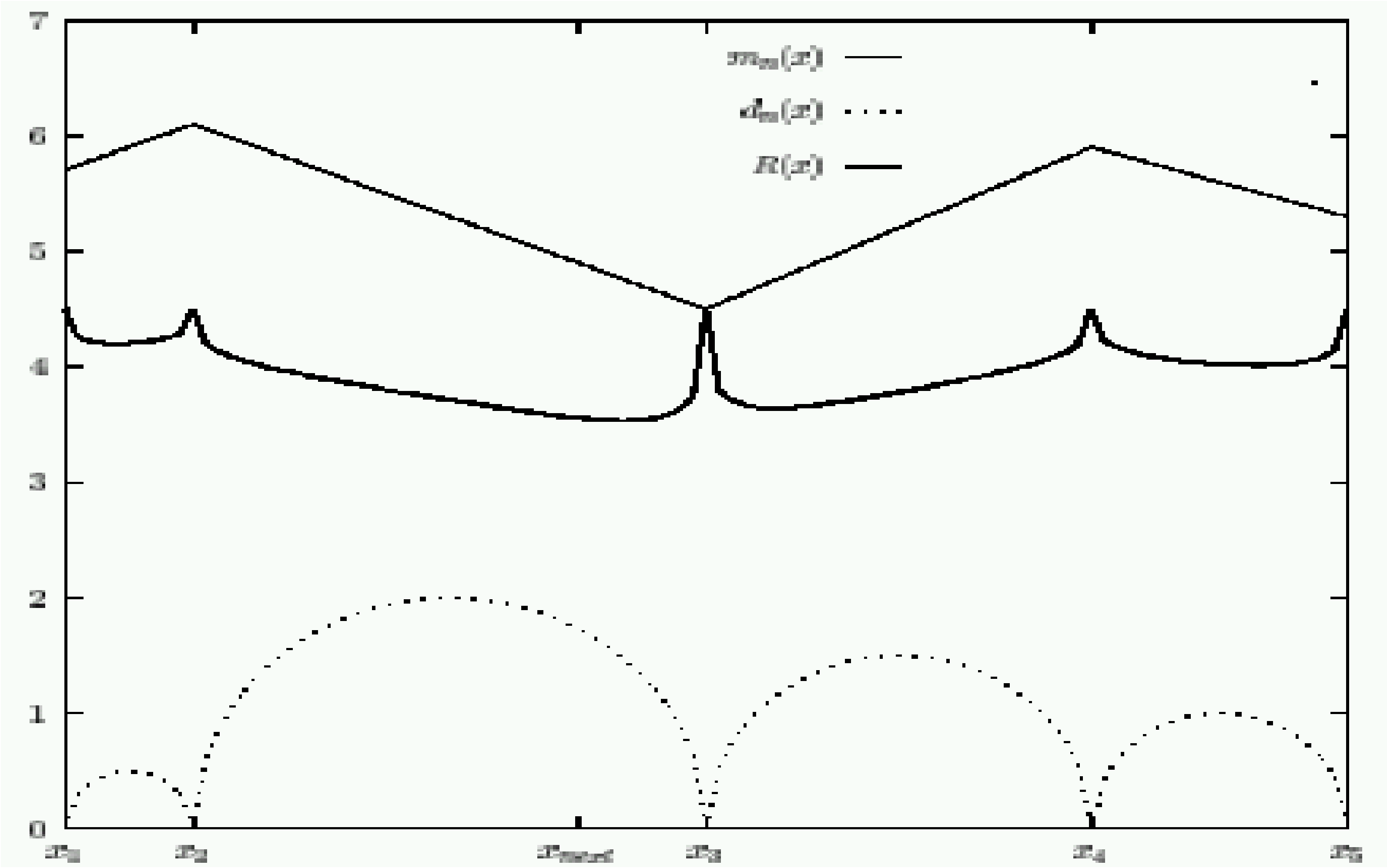
Building nearly-uniform grids

In multi-dimensional global optimization problems building exactly uniform grids is difficult.
Then "LPtau" or "Monte Carlo" approximations are used.
LPtau grids provide nearly-uniform coordinate projections.
Monte Carlo grids generate coordinates $x_i, i = 1, ..., n$ independently by uniform probability distributions.

Bayesian methods

Uniform grids minimize maximal deviation by using
$T = C2^{mn}$ of computing time, where
n is number of variables,
m defines the accuracy by the condition $\epsilon \leq 2^{-m}$.
This is expensive if m or n are large.
Then Bayesian methods are applied.
Bayesian methods minimize expected deviation for a given iteration number.
Defining expected deviation statistical models of objective functions are needed.
Simple one-dimensional example is the Wiener model.

Fig. 2. Wiener model



Wiener model optimization

In Fig. 2. $m_n(x)$ is the conditional mean,
$d_n(x)$ is the conditional variance,
$R(x)$ is the risk function,
Bayesian method minimizes the risk function

$$x^{n+1} = arg \min_x R(x). \qquad (216)$$

Risk function

In the Wiener model the risk function

$$R(x) = \frac{1}{\sqrt{(2\pi d_n(x))}} \int_{-\infty}^{+\infty} \min\left(c_n, y_x\right) e^{-\frac{(y_x - m_n(x))^2}{d_n(x)}} dy_x$$

Here
$y_x = f(x)$,
$c_n = \min_i f(x^i) - \epsilon,\ \epsilon > 0$.
Note that
$R(x) = c_n$, if $x = x^i$
since
$d_n(x^i) = 0$.

Coordinate optimization

Wiener model is applied for coordinate optimization when $m > 1$,

$$x^1 = arg\min_{x_1} f(x^0) \tag{217}$$

$$x^2 = arg\min_{x_2} f(x^1) \tag{218}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x^m = arg\min_{x_m} f(x^{m-1}) \tag{219}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Here results depend on initial points x^0.
However the coordinate optimization is a convenient tool of visualization by one-dimensional projections. Coordinate optimization by Wiener model converges if

$$f(x) = \sum_i f_i(x_i) \ or \ f(x) = \prod_i f_i(x) \tag{220}$$

Multi-dimensional Bayesian method

The point of next observation (calculation of $f(x)$):

$$x^{n+1} = arg\min_x R(x) \tag{221}$$

$$R(x) = y_{0n} - \min_i \frac{||x - x_i||^2}{f(x_i) - c_n}, \tag{222}$$

$$c_n = \min_i f(x^i) - \epsilon,\ \epsilon > 0. \tag{223}$$

When n is large

$$d^*/d_a = (\frac{f_a - f^* + \epsilon}{\epsilon})^{1/2}$$

Here d^* density of observations in the vicinity of x^*,
f^* average value $f(x)$ around x^*,
d_A average density of observations, f_A average values of $f(x)$.

Vector optimization

Objective is to maximize vector-function

$$f(x) = (f_i(x),\ i = 1, ..., m). \tag{224}$$

Pareto optimum is the set X^*.
$x^* \in X^*$, if there are no such x that

$$f_i(x) \geq f_i(x^*),\ \forall i \tag{225}$$

$$f_j(x) > f_j(x^*),\ \exists j \tag{226}$$

Scalarization

By weights

$$x(c) = arg\max_x \sum_i c_i f_i(x),\ c_i > 0. \quad (227)$$

Here $x(c) \in X^*$.
'bf By constraints

$$x(b) = arg\max_x f_1(x) \quad (228)$$

$$f_i(x) \geq b_i,\ i = 2, ..., m. \quad (229)$$

Here $x(b) \in X^*$, if $x(b)$ is unique,
otherwise, non-Pareto points are possible.

Simulated Annealing (SA)

Simulated Annealing (SA) is the most popular global optimization method Denote

$$h_j = f(x^j) - f(x^{j-1}), \tag{230}$$

if $h_j \geq 0$, then go to x^j
if $h_j < 0$, then go to x^j with probability

$$r_j = \begin{cases} e^{\frac{h_j}{x/\ln(1+j)}}, & \text{if } h_j < 0, \\ 1, & \text{otherwise} \end{cases} \tag{231}$$

SA is very simple and convenient for theoretical analysis. Efficiency of SA is increased by optimization of parameters, for example by optimization of x for given family of objective functions $f(x)$ at fixed number of iterations. Here the Bayesian approach is useful.

Sawmil problem

Objective is to minimize waste wile sawing $i = 1, ..., m$ planks from $j = 1, ..., n$ logs.
All dimensions are given.
Denote by h_{ij} priority rule used defining what plank i to saw and from which log j.
The results depends on goodness of heuristic h_{ij} and on the efficiency of algorithms providing geometric constraints.
Here heuristics are not simple. Geometric constrains keeping planks inside logs are complicated.

Formalization of sawmil problem

$$\min_{y}\ v(y),\ y=(y_{ijk}), \tag{232}$$

$$i=1,...,m,\ j=1,...,n,\ k=1,...,K,$$

$$a_i y_{ijk} \le d_{ijk}.$$

Here $v(y)$ is the waste calculated as difference between the volumes of logs and planks, a_i is thickness of plank i, d_{ijk} is thickness of log j at place k where plank i sawed.

$$y_{ijk}=\begin{cases}1, & \text{jei } i\in(j,k),\\ 0, & \text{otherwise}\end{cases} \tag{233}$$

where i number of plank, j is number of log, k defines the place where plank i is sawed from log j, $i\in(j,k)$ means that plank i is in log j at the place k.

Fig. 3. Complete cut

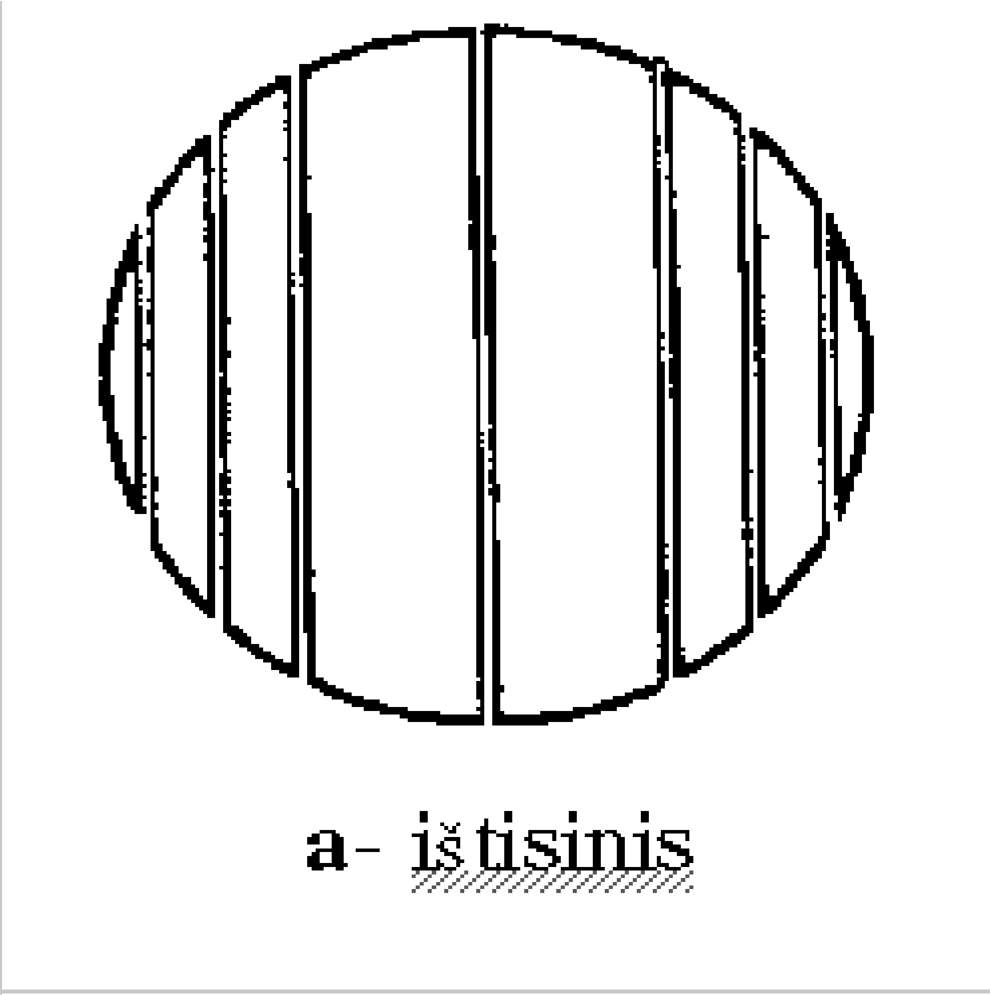
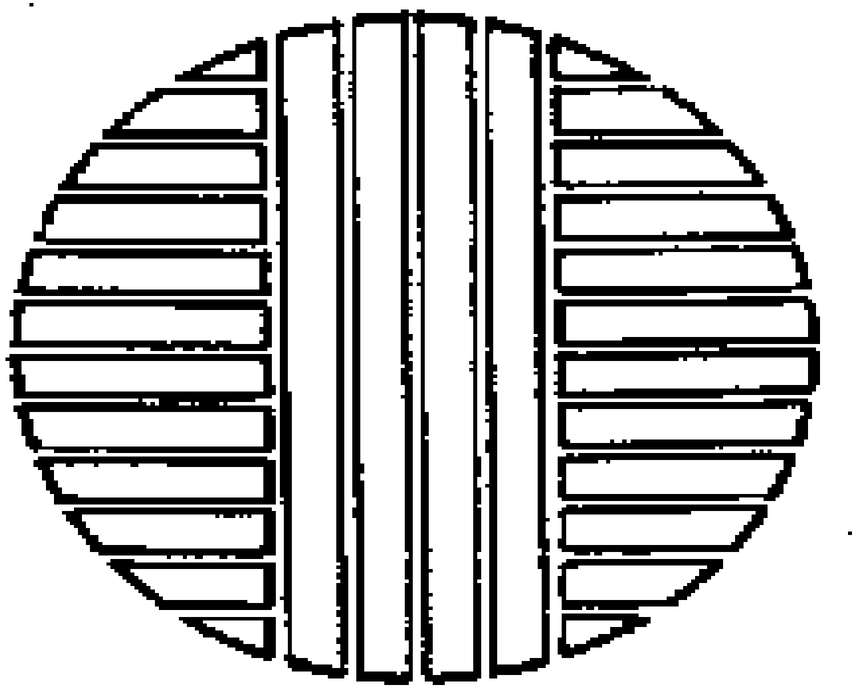


Fig. 4. Segment cut



c – ištisinis segmentinis

Completing trains

The objective is to minimize the number of train $n = n(y)$ for m cars

$$\min_{y} \; n(y), \tag{234}$$

$$\sum_{i=1}^{m} a_i y_{ij} \leq A_j, \tag{235}$$

$$\sum_{i=1}^{m} b_i y_{ij} \leq B_j. \tag{236}$$

Here

$$y_{ij} = \begin{cases} 1, & \text{jei } i \in j, \\ 0, & \text{otherwise.} \end{cases} \tag{237}$$

Completing trains, notation

In the formula of completing trains: a_i is weight of car i,
A_j is maximal weight of train j,
b_i is length of car i,
B_j is maximal length of train j,
$i \in j$ means that the car i is in the train j.

Theory of games and applications

In the optimization part of this course the models of games and markets are regarded as examples of optimization problems.
In the game-theoretical part basic elements of theory of games and markets are explained and some additional problems are introduced.

Simplest game, "Toss-up"

Here two players and two moves.
Payoff matrix of the first player;

$$u(i,j) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \tag{238}$$

where $i = 1, 2$ are moves of the first player,
and $j = 1, 2$ are moves of the second player.
Payoff matrix of the second player

$$v(i,j) = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \tag{239}$$

Here $v(i,j) = -u(i,j)$, that is a zero-sum game.

Byesian game, "Toss-up"

Here two players,
first is a person, second is the "nature"
and two moves.
Payoff matrix of the first player;

$$u(i,j) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \tag{240}$$

where $i = 1, 2$ are moves of the first player,
and $j = 1, 2$ are moves of the 'nature'.
In Bayesian game nature uses mixed strategy,
assume that $p(1) = 1/2 + \epsilon,\ p(2) = 1 - p(1)$
then optimal strategy of the first player is 1-st row

Pure and mixed strategies

Moves made directly are pure strategies. Moves made by random procedures are mixed strategies.
Expected payoffs are players objectives if mixed strategies are used.

$$\begin{aligned} U(x,y) &= x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 \\ V(x,y) &= -x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2 \end{aligned} \tag{241}$$

where
$0 \le x_i \le 1,\ i = 1,2$ is mixed strategy of the first player (probability of making the move i),
and y_i is mixed strategy of the second player
From here

$$\begin{aligned} U(x,y) &= 4x_1y_1 - 2x_1 - 2y_1 + 1 \\ V(x,y) &= -4x_1y_1 + 2x_1 + 2y_1 - 1 \end{aligned} \tag{242}$$

Equilibrium strategies

Equilibrium are strategies with no incentives for change. In the "Toss-up" game there is no equilibrium by pure strategies. But there exists an equilibrium by mixed strategies

$$x_i = y_i = 0.5 \tag{243}$$

$$\begin{aligned} U(x_1 = 1/2, y_1 = 1/2) = 0, \\ V(x_1 = 1/2, y_1 = 1/2) = 0 \\ x_2 = 1 - x_1, y_2 = 1 - y_1. \end{aligned} \tag{244}$$

Here mixed strategies $x_i = y_i = 0.5$ are generated by tossing a coin. We see that

$$\begin{aligned} U(x_1 = 1/2 + \epsilon, y_1 = 1/2) < 0, \\ V(x_1 = 1/2, y_1 = 1/2 + \epsilon) < 0, \end{aligned} \tag{245}$$

if $\epsilon \neq 0$.

Bimatrix games

In bimatrix games $u(i, j) \neq -v(i, j)$
Simple example is asymmetric version of the "Toss-up" game

$$u(i, j) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \quad (246)$$

$$v(i, j) = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \quad (247)$$

Strategies of bimatrix game

Expected payoff

$$U(x,y) = x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2,$$
$$V(x,y) = -x_1y_1 + x_1y_2 + x_2y_1, \quad (248)$$

or

$$U(x,y) = 4x_1y_1 - 2x_1 - 2y_1 + 1,$$
$$V(x,y) = -3x_1y_1 + x_1 + y_1. \quad (249)$$

Here the equilibrium

$$y_1 = y_2 = 1/2, \quad (250)$$
$$x_1 = 1/3, x_2 = 2/3, \quad (251)$$
$$u^* = U(x_1 = 1/3, y_1 = 1/2) = 0, \quad (252)$$
$$v^* = V(x_1 = 1/3, y_1 = 1/2) = 1/3, \quad (253)$$

Equalizing expected payoffs

Denote

$$U(1,y) = y_1 - y_2, U(2,y) = -y_1 + y_2, \quad (254)$$

$$V(x,1) = -x_1 + x_2, V(x,2) = x_1, \quad (255)$$

where $U(i,y)$ expected payoff of the first player using the move i if the second uses mixed strategy y, expected payoff of the second player $V(x,j)$ is defined similarely
Strategies x, y can be defined using linear programming

$$U(1,y) = U, U(2,y) = U,$$
$$V(x,1) = V, V(x,2) = V. \quad (256)$$

where $0 \le x_i \le 1, 0 \le y_i \le 1, x_1 + x_2 = 1, y_1 + y_2 = 1$. The solution if exists satisfies equilibrium condition.

Equilibrium in pure strategies

Equilibrium in pure strategies of of bimatrix games can be found by comparing all pairs (i, j) of pure strategies. For example:

$$u(i,j) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \tag{257}$$

$$v(i,j) = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} \tag{258}$$

Equilibrium strategies are $(i = 2, j = 2)$ and payoffs are $u(2,2) = 1, v(2,2) = 2$.
Note that here is no solution of the linear programming problem equalizing expected payoffs.

Prisoners' dilema

1 yes, 2 no rows controlled 1-st prisoner
columns controlled 2-cond prisoner

$$u(i,j) = \begin{vmatrix} 3 & 1 \\ 10 & 2 \end{vmatrix} \tag{259}$$

$$v(i,j) = \begin{vmatrix} 3 & 10 \\ 1 & 2 \end{vmatrix} \tag{260}$$

Here
Equilibrium (1,1)
Pareto (2,2)(2,1)(1,2)

Family problem

1 opera, 2 soccer
rows controlled by husband
columns controlled by wife

$$u(i,j) = \begin{vmatrix} 10 & 0 \\ 0 & 20 \end{vmatrix} \tag{261}$$

$$v(i,j) = \begin{vmatrix} 20 & 0 \\ 0 & 10 \end{vmatrix} \tag{262}$$

Here
Equilibrium (1,1)(2,2)
Pareto (1,1)(2,2)

The deal

If $v(i, j) \neq -u(i, j)$,
then deal is possible.
For example inspector can make deal with poacher to devide the pray.
The solution is Nash deal.
Nash deal depends on a set D of feasible payoff partitions and on the "no-deal" payoffs

$$u^* = max_x\ min_y U(x, y),$$
$$v^* = max_x\ min_y V(x, y). \tag{263}$$

Here
u^* is maximal guaranteed payoff of the first player
and v^* is maximal guarantee payoff of the second player.
The deal is $(\bar{u}, \bar{v})$, where $\bar{u}$ is payoff of the first player and $\bar{v}$ is payoff of second player.

Nash axioms

1. $(\bar{u}, \bar{v}) \geq (u^*, v^*)$,
2. $(\bar{u}, \bar{v}) \in D$,
3. if $(u, v) \in D$ ir $(u, v) \geq (\bar{u}, \bar{v})$, then $(u, v) = (\bar{u}, \bar{v})$,
4. if $(\bar{u}, \bar{v}) \in T \subset D$ ir $(\bar{u}, \bar{v}) = \phi(D, u^*, v^*)$, then $(\bar{u}, \bar{v}) = \phi(T, u^*, v^*)$,
5. if a set D' is obtained from the set D by these equalities $u' = a_1 u + b_1,\ v' = a_2 v + b_2$, then, from $\phi(D, u^*, v^*) = (\bar{u}, \bar{v})$ follows that $\phi(D', a_1 u + b_1,\ a_2 v + b_2) = (a_1 u + b,\ a_2 v + b_2)$,
6. if $(u, v) \in D \Leftrightarrow (v, u) \in D$, $u^* = v^*$ ir $\phi(D, u^*, v^*) = (\bar{u}, \bar{v})$, then $\bar{u} = \bar{v}$.

Nash deal

If Nash axioms are true then exists an unique deal function $\phi(D, u^*, v^*) = (\bar{u}, \bar{v})$
If there is such pair $(u, v) \in D$ that $u > u^*,\ v > v^*$, then the deal is

$$(\bar{u}, \bar{v}) = arg\ max_{u,v}(u - u^*)(v - v^*), \tag{264}$$

where

$$(u, v) \in D,\ u \geq u^*,\ v \geq v^* \tag{265}$$

If the feasible payoff is limited by c then

$$D = \{(u, v) : u + v \leq c\}. \tag{266}$$

Here the deal

$$(\bar{u}, \bar{v}) = ((c + u^* - v^*)/2,\ (c + v^* - u^*)/2) \tag{267}$$

Nash deal, simple example

In simple example of bimatrix game
$(u^*, v^*) = (0, 1/3)$.
If feasible payoff $u + v$ is not limited then the deal
$(\bar{u}, \bar{v}) = (1, 1)$.
If feasible payoff is limited: $u + v \leq c = 1$,
then the deal
$(\bar{u}, \bar{v}) = (1/3, 2/3)$.
If the payoff limit is $c = 1/3$
then a part of the first player in the deal is zero
$(\bar{u}, \bar{v}) = (0, 1/3)$
Therefore the first player makes no deal because no-deal payoff is high enough
$u^* = \bar{u}$.

Nash deal, inspector's example

In the inspector-poacher deal
feasible payoff is limited by $c = 1$.
Guaranteed payoffs are
$u^* = 2/9, \ v^* = 5/9$.
Then the Nash deal
$(\bar{u}, \bar{v}) = (1/3, 2/3)$.
This deal is stable since
$(1/3, 2/3) > (2/9, 5/9)$
The deal can be prevented prevented if inspector's
expected deal penalty
$B > \bar{u} - u^* = 1/9$.
Here $B = p_b \, b$ where p_b is probability of deal penalty b .
If the penalty $b = 1$ is equal to price of pray
then the penalty probability should be $p_b > 1/9$.

Nash deal, strike example

Here $x \in [0, a]$ is employer's pure strategy that means to pay salary x, where a is the employer's income.
Employee's strategies are

$$y = \begin{cases} 0, & \text{strike} \\ 1, & \text{work} \end{cases} \tag{268}$$

Employer's payoff

$$u(x, y) = \begin{cases} a - x, & \text{if } y = 1 \\ -x, & \text{otherwise.} \end{cases} \tag{269}$$

Employee's payoff

$$v(x, y) = \begin{cases} x, & \text{if } x > 0 \\ -b, & \text{if } x = 0. \end{cases} \tag{270}$$

Strike example, payoffs

Here guaranteed payoffs
$u^* = 0,\ v^* = -b$,
where b define the employee's "zero-income" loss.
The feasible payoff is limited by $c = a$.
If $b < a$, then the deal

$$(\bar{u}, \bar{v}) = ((a+b)/2,\ (a-b)/2). \tag{271}$$

Equilibrium, formal definition

Notation:
m is number of players,
$y_j, j = 1, ..., m$ are strategies of players j,
$u_i = u_i(y_j,\ j = 1, ..., m),\ i = 1, ..., m$ is expected payoff of the player i that depends on the strategies of all players.
$y_j^0,\ j = 1, ..., m$ is a "contract" strategy of player j, and $U_j^0\ j$ is the expected "contract" profit . Suppose that a player brakes the contract only if expects larger profit

$$U_j^1 > U_j^0, \tag{272}$$

$$U_j^1 = max_{y_j} u_j(y_1^0, ..., y_j, ..., y_m^0), j = 1..., m. \tag{273}$$

Thus a contract $y^0 = (y_j^0, j = 1, ..., m)$ is Nash equilibrium if

$$\Delta U = \sum_j (U_j^1 - U_j^0) = 0. \tag{274}$$

Equilibrium, sufficient conditions

Equilibrium exists if
1. Expected payoffs $U_i(x_j, j = 1, .., m)$ are convex functions of strategies x_j
2. Sets X_j of feasible strategies x_j are convex.
In the "Toss-up" game the set of two pure strategies $x_1 = 0$ and $x_2 = 1$ is not convex and no equilibrium exists.
A set of mixed strategies $x \in [0, 1]$ is convex, so the equilibrium exists.

Contracting operator

Stability of equilibrium depends on operator T: $y^{n+1} = T(y^n)$ transforming the "contract" vector , $y^0 = (y^0_j,\ j = 1, ..., m)$ into "no-contract" vector y^1 by maximization of the expected profit

$$y^1_j = arg\ max_{y_j} u_j(y^0_1, ..., y_j, ..., y^0_m), j = 1..., m. \tag{275}$$

under the assumption that competitors honour the contract. Operator T is contracting if

$$\frac{||T(yn+1) - T(yn)||}{||yn+1-yn||} \leq \rho < 1 \tag{276}$$

Equilibrium is stable if T is contracting.

Fig.5. Fixed point exists

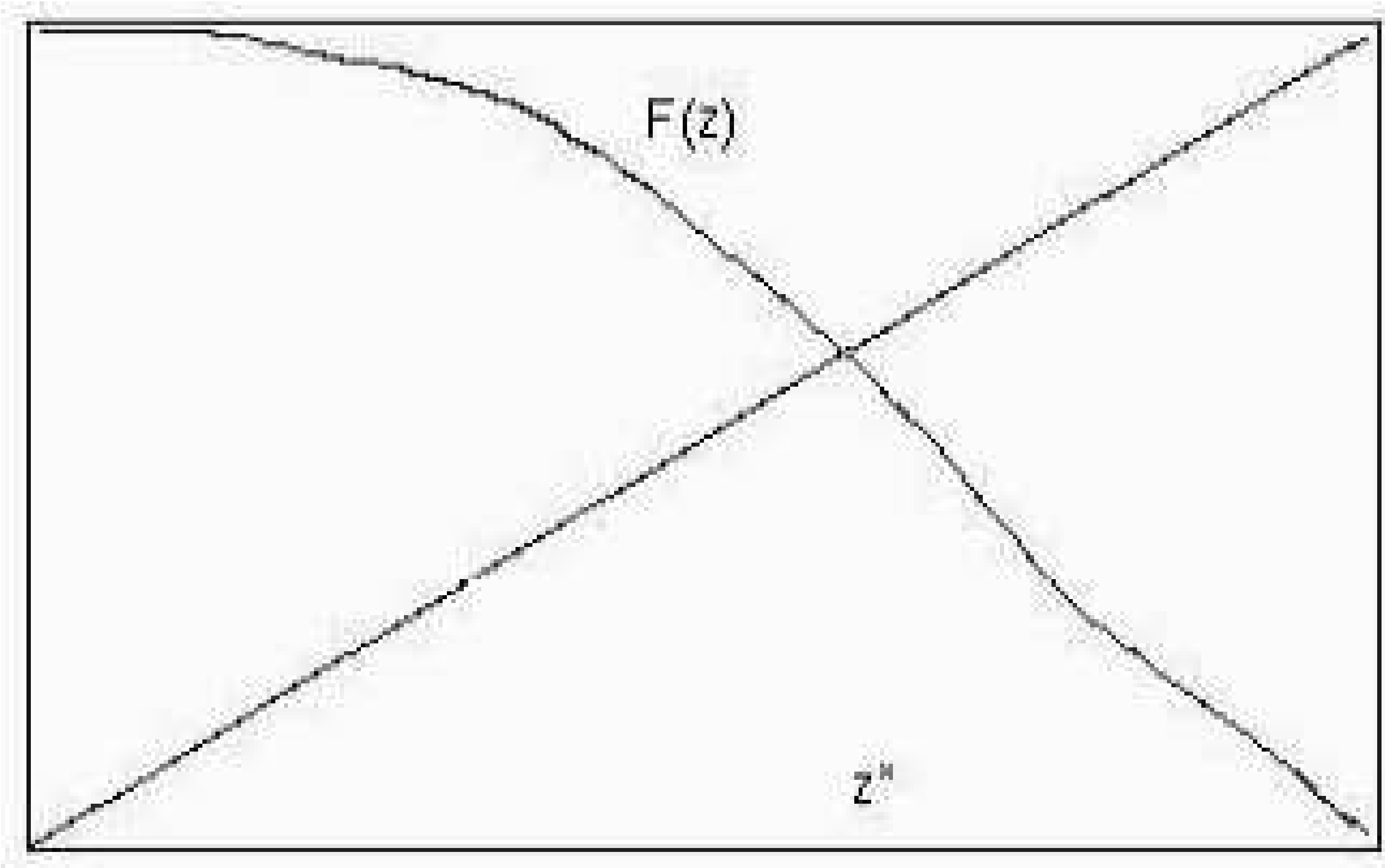


Fig. 6. No fixed point

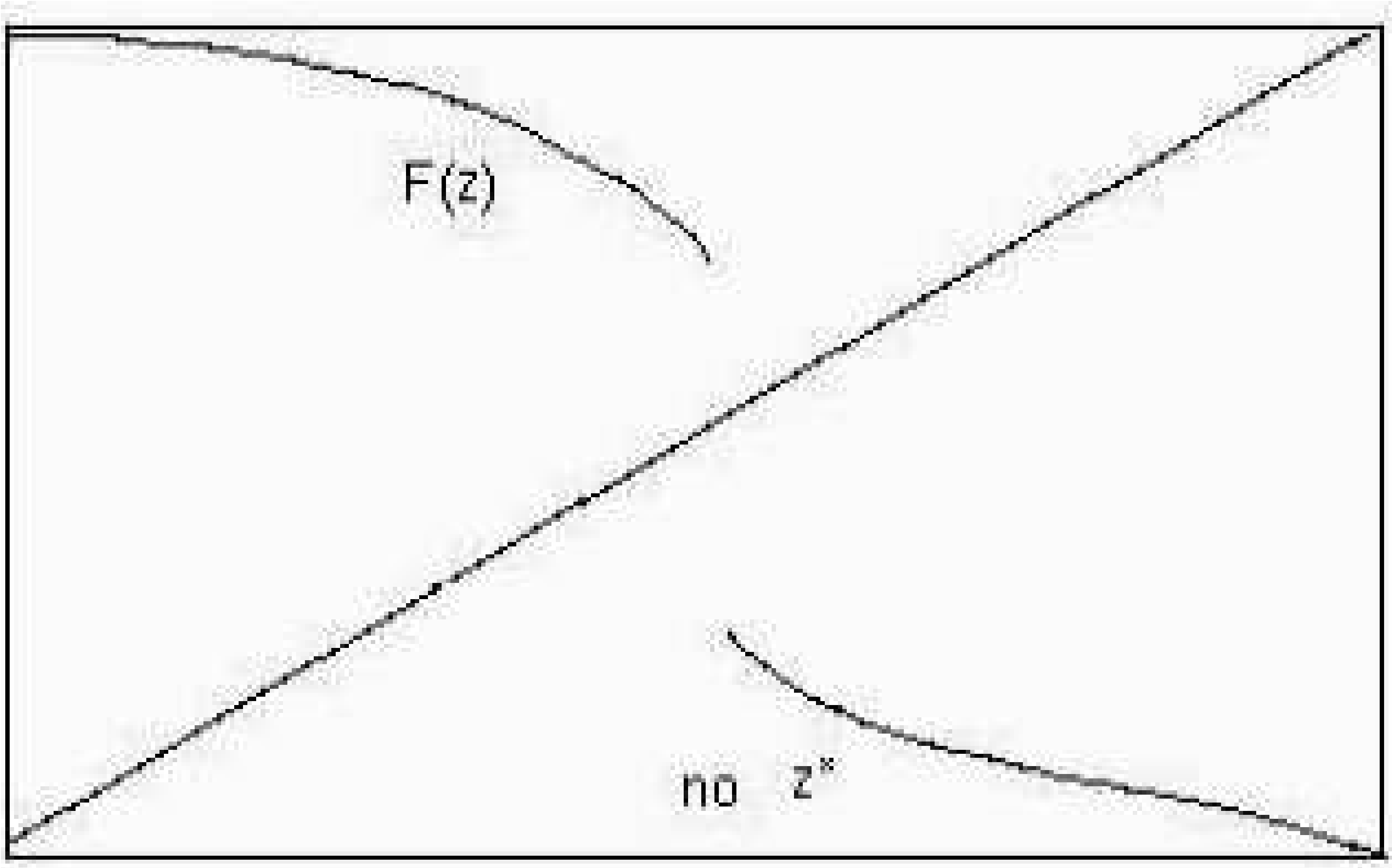


Fig.7. Several fixed points

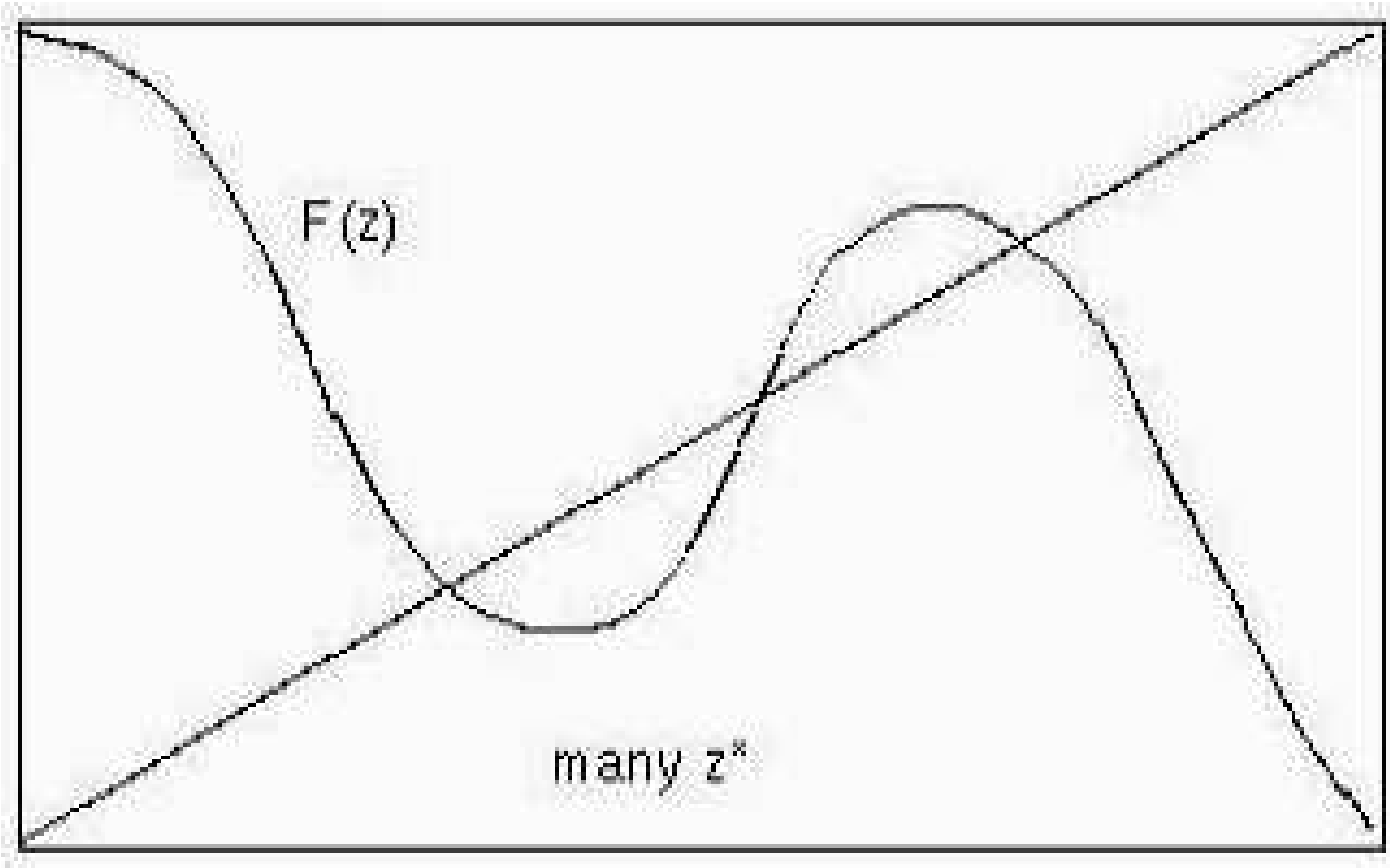


Fig. 8. Stable fixed point

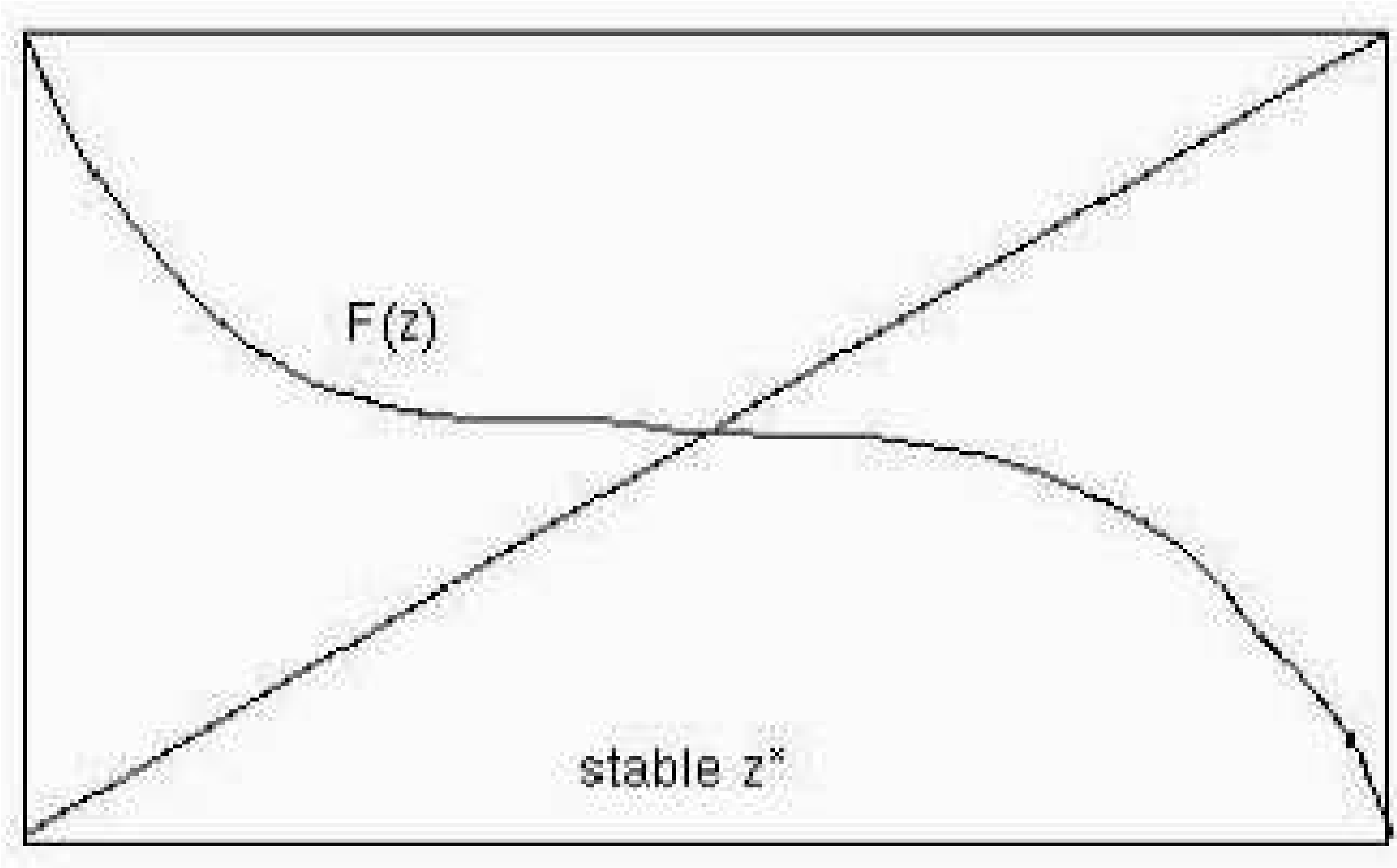


Fig. 9. Unstable fixed point

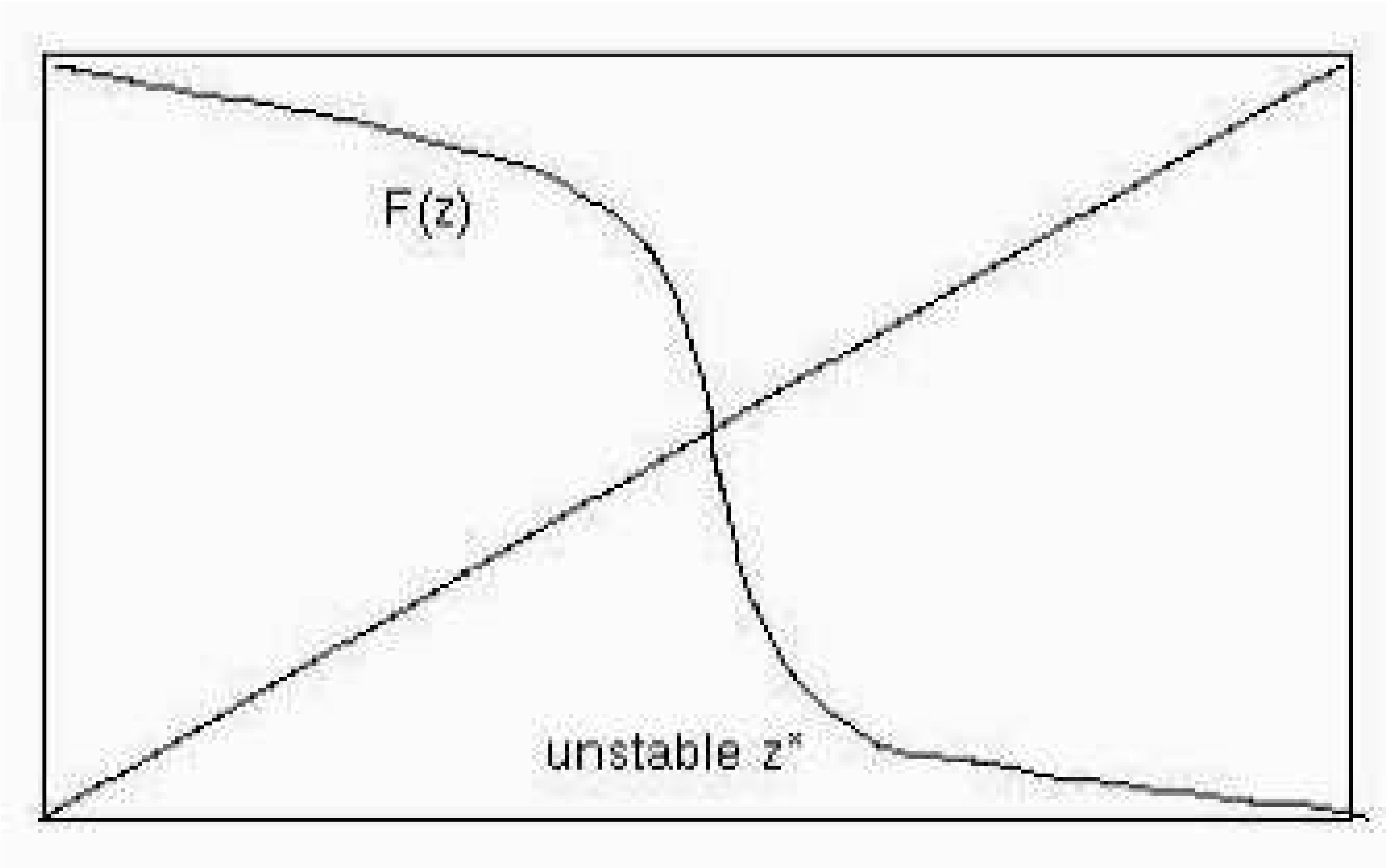
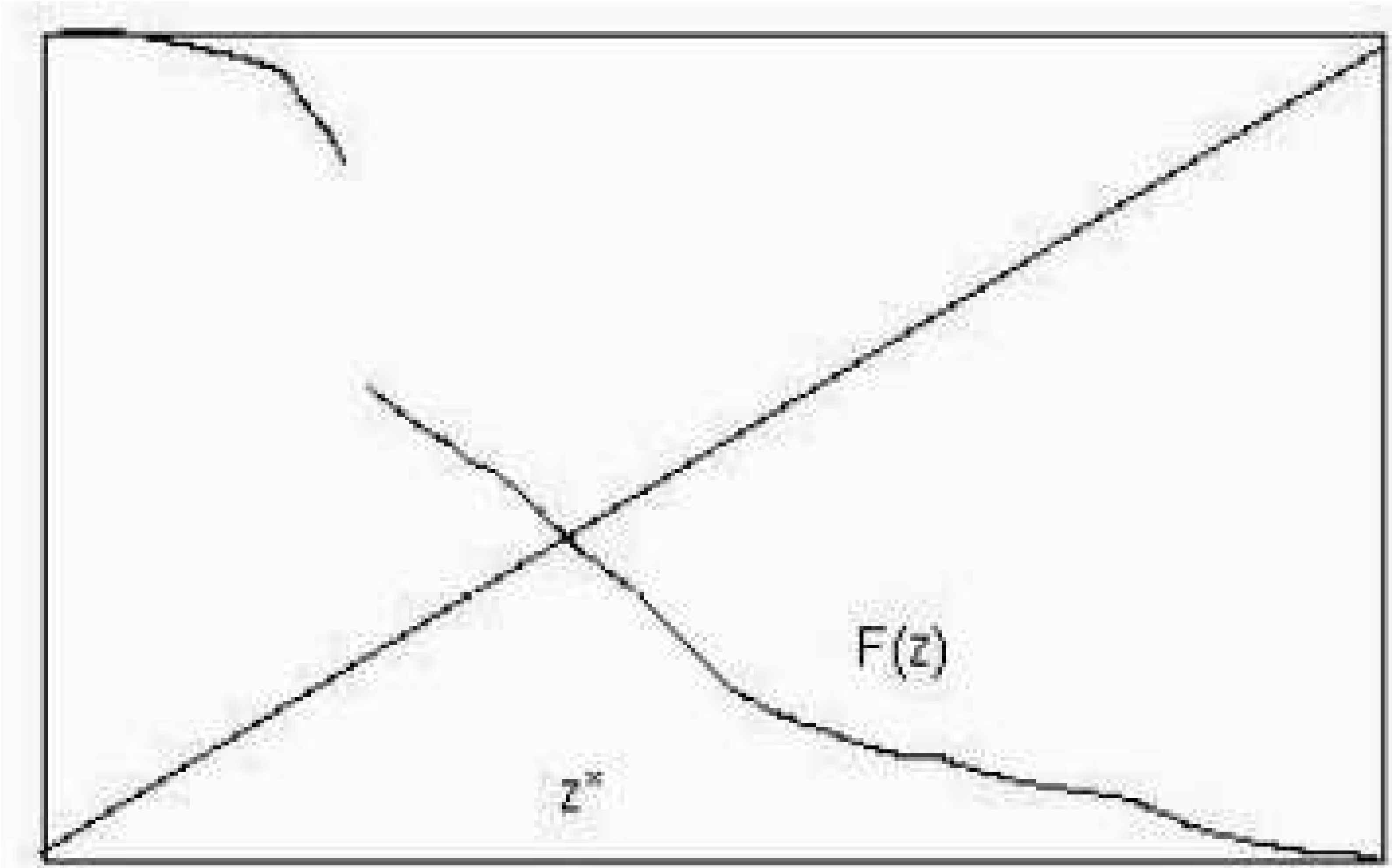


Fig. 10. Discontinuous example



Cooperative games

Stability of coalitions is important if the number of players $m > 2$.

Stability of a coalition depends on the guaranteed payoff of the coalition and on the partition of this payoff.

A coalition is stable if no changes will provide greater part of guaranteed payoff.

The guaranteed payoff of a coalition is defined by the characteristic function.

Characteristic function

Notation:
S is a set of all players.
A subset $s \subset S$ is coalition.
Maximal guaranteed payoff of coalition s

$$v(s) = max_x \; min_y \; u_s(x, y). \tag{277}$$

Here $x = x(s)$ a strategy of coalition s,
$y = y(S \setminus s)$, a strategy of coalition of remaining players,
$v(s)$ is characteristic function.
A game v is fixed sum game if

$$v(s) + v(S \setminus s) = v(S). \tag{278}$$

A game is essential if

$$\sum_{i \in S} v(i) < v(S). \tag{279}$$

Cooperative game, "Three Boys"

Consider three players $i = 1, 2, 3$ and three coalitions $S_j = 1, 2, 3$, where $s_1 = \{1, 2\}$, $s_2 = \{1, 3\}$, $s_3 = \{2, 3\}$. Table shows parts of player payoffs in different coalitions

$$u(i,j) = \begin{vmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{vmatrix} \qquad (280)$$

Here the characteristic function

$$v(s) = \begin{cases} 2, & \text{if } |s| = 2 \\ -2, & \text{if } |s| = 1. \end{cases} \qquad (281)$$

where $|s|$ is a number of players in coalition s.

Game properties, "Three Boys"

This is fixed sum game since

$$v(s) + v(S \setminus s) = v(S) = 0. \tag{282}$$

The game is essential because

$$\sum_{i \in S} v(i) = -6 < v(S) = 0. \tag{283}$$

Cooperative game, "Joint-Stock"

Consider stock holders $i = 1, 2, 3, 4$ and their coalitions s_j. Numbers of stocks $g_1 = 10, g_2 = 20, g_3 = 30, g_4 = 40$. Coalition s_j is wining if $G_j > 50$, where $G_j = \sum_{i \in s_j} g_i$. Here the characteristic function

$$v(s_j) = \begin{cases} 1, & if \ G_j > 50 \\ 0, & if \ G_j \leq 50. \end{cases} \tag{284}$$

v is a fixed sum game since

$$v(s) + v(S \setminus s) = v(S) = 0. \tag{285}$$

v is essential game because

$$\sum_{i \in S} v(i) = 0 < v(S) = 1. \tag{286}$$

Payoff partition

Notation:
z_i is a part of player i in the payoff partition $z = (z_1, ..., z_m)$, A partition z is feasible if

$$\sum_{i \in S} z_i = v(S) \tag{287}$$

$$z_i \geq v(i) \tag{288}$$

A partition z dominate partition w by coalition s
$z \succ_s w$,
if

$$\sum_{i \in s} z_i \leq v(s) \tag{289}$$

$$z_i > w_i, \; i \in s \tag{290}$$

If there is such a coalition s then we say $z \succ w$

Core of game

The core $C(v)$ of the game v is a set of all stable partitions. A partition is stable if no coalition can offer better partition. Core provides stability of coalitions since there are no incentives for changes. In practice it means political and economic stability.
However there is no core in essential fixed sum games: $C(v) = \emptyset$ if

$$\sum_{i \in S} v(i) < v(S),$$
$$v(s) + v(S \setminus s) = v(S).$$

Such are examples 1 and 2.
In competition models "Nash" and "Walras" the core $C(v)$ may exist because they both are not fixed sum games

Shapley vector

If no core exists convenient tool of partition is Shapley vector. Shapley partition can be stable too if all the payers understand and agree with the Shapley conditions:

1. $\sum_{i \in s} \phi_i[s] = v(s)$,
   where $\phi_i[s]$ is Shapley partition,
2. $\phi_{\pi(i)}[\pi v] = \phi_i[v]$, where π is permutation of players,
3. $\phi_i[u+v] = \phi_i[u] + \phi_i[v]$, where u and v are two games.

If these conditions are true then there exists the unique Shapley partition:

$$\phi_i[v] = \sum_{s \subset S,\ i \in s,} (|s|-1)!(|S|-|s|)!/|S|! \\ (v(s) - v(s \setminus \{i\}), \qquad (291)$$

Shapley vector, relevant coalitions

For a player i relevant are only those coalitions S_i that wins with the player i and loses without this player. Thus a player i can pretend for a part of payoff of coalition S_i

Shapley partition, "Three Boys"

In the "Three boys" example:
$S_1 = \{\{s_1, s_2\}, \{s_1, s_3\}, S_2 = \{s_2, s_3\}, \{s_1, s_2\}, S_3 = \{s_2, s_3\}, \{s_1, s_3\}\}$.
Here Shapley partition

$$\phi_i[v] = (2-1)!(3-2)!/3! + (2-1)!(3-2)!/3! = 1/3. \tag{292}$$

Shapley partition, "Joint-Stock

In the "Joint Stock" example:
$S_1 = \{1,2,3\}$, $S_2 = \{\{1,2,3\},\{2,4\},\{1,2,4\}\}$,
$S_3 = \{\{1,2,3\},\{3,4\},\{1,3,4\}\}$,
$S_4 = \{\{2,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\{3,4\}\}$.

$$\phi_1[v] = (2-1)!(4-1)!/4! = 1/12,$$
$$\phi_2[v] = 2(3-1)!(4-3)!/4! + (2-1)!(4-2)!/4! = 3/12,$$
$$\phi_3[v] = 2(3-1)!(4-3)!/4! + (2-1)!(4-2)!/4! = 3/12,$$
$$\phi_4[v] = 3(3-1)!(4-3)!/4! + 2(2-1)!(4-2)!/4! = 5/12.$$

Stabilization of partitions

Payoff partitions belonging to the core of game $C(v)$ are stable without conditions.
Shapley partitions are stable if all the players understand and agree with Shapley conditions.
That means that players predict correctly the reaction of their partners and corresponding consequences. That is not always a practical assumption.
Practically partitions can be stabilized by penalties for "bad" behavior and bonuses for "good" behavior.
That transforms fixed sum game into non-fixed sum game with stable core $C(v)$.

Stabilization of game, "Three Boys"

Introduce in the "Tree Boys" example a "Unity Bonus"-additional payoff +1 for each "boy" if all three unites to form a coalition $s_4 = \{1, 2, 3\}$

$$u(i,j) = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{vmatrix}. \tag{293}$$

Then the characteristic function

$$v(s) = \begin{cases} 2, & if \text{ |s|=2} \\ -2, & if \text{ |s|=1} \\ 3, & if \text{ |s|=3} \end{cases} \tag{294}$$

Core $C(v) = (1, 1, 1)$ implements the coalition $s_4 = \{1, 2, 3\}$.

$$v(s) + v(S \setminus s) \neq v(S) = 3, \ |s| < 3. \tag{295}$$

Stabilization of game, "Joint-Stock"

If we introduce in the "Joint Stock" game a penalty for coalitions for braeking the rules of proportional partitions then the characteristic function

$$v(s_j) = \begin{cases} 1, & if\ G_j > 50\ and\ z_i \in Z,\ i \in s_j \\ 0, & if\ G_j > 50\ and\ z_i \bar{\in} Z,\ i \in s_j \\ -1, & if\ G_j \leq 50\ and\ z_i \bar{\in} Z,\ i \in s_j. \end{cases}$$

Here Z is a partition of payoff in proportion to stock number. Here exists core of game $C(v) = (10, 20, 30, 40)$ implementing the coalition $s = \{1, 2, 3, 4\}$.

AR-ABS models

This is a version of AR (Auto-Regression) model

$$w_t = \sum_{i=1}^{p} a_i w_{t-i} + \epsilon_t, \tag{296}$$

where
w_t prediction for tomorrow
w_{t-1} observed value today,
ϵ_t random unpredictable variable tomorrow,
a_i coefficients of "importance"
defined by minimization of residual

$$f(x) = \sum_{t=1}^{T} |\epsilon_t|. \tag{297}$$

Solving AR-ABS model

We minimize the residual by linear programming

$$\min_{a,u} \sum_{t=1}^{T} u_t \tag{298}$$

$$u_t \geq \epsilon_t, \ t = 1, ..., T, \tag{299}$$

$$u_t \geq -\epsilon_t \ t = 1, ..., T, \tag{300}$$

$$a_i = a_i^1 - a_i^2, \ a_i^k \geq 0, \ k = 1, 2, \ i = 1, ..., p. \tag{301}$$

Last slide

This is the last slide. Do you want to go to the second one?